New asymptotic expansion for the $\Gamma(x)$ function

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Abstract

Using a series transformation, Stirling-De Moivre asymptotic series approximation to the Gamma function is converted into a new one with better convergence properties. The new formula is compared with those of Stirling, Laplace and Ramanujan for real arguments greater than 0.5 and turns out to be, for equal number of 'correction' terms, numerically superior to all of them. As a side benefit, a closed-form approximation has turned up during the analysis which is about as good as 3rd order Stirling's (maximum relative error smaller than 1e-10 for real arguments greater or equal to 10). Note: this article is an extended version of an older one [7] to which it adds the estimate of the remainder.

1 Introduction

1.1 The main result

In this paper, as we claimed in the abstract, we present a new asymptotic expansion for the gamma function for real arguments greater or equal than one. We give an explicit formula for the coefficients in the series and estimate the error. After the proof, we compared our new formula with some classical results. The numerical comparison shows that for equal number of 'correction' terms the new formula is the most accurate and it is highly recommended for computing the Gamma function for large real arguments.

Theorem 1. *Let $x \geq 1$, then for every $n \geq 1$, the following expression holds:*

$$\Gamma(x) = \left(\frac{x}{e}\left(\sum_{k=0}^{n-1}\frac{G_k}{x^{2k}} + R_n(x)\right)\right)^x \sqrt{\frac{2\pi}{x}}, \tag{1.1}$$

where $R_n(x) = \mathcal{O}\left(\frac{1}{x^{2n}}\right)$ and the G_k coefficients are given by

$$G_k = \sum_{\substack{m_1,m_2,\ldots,m_k \geq 0 \\ 2m_1+4m_2+\ldots+2km_k=2k}} \prod_{r=1}^{k}\frac{1}{m_r!}\left(\frac{B_{2r}}{2r(2r-1)}\right)^{m_r}, \quad G_0 = 1, \tag{1.2}$$

where B_r denotes the r^{th} Bernoulli number [4]. Moreover, if $x \geq n+1$, then

$$|R_n(x)| \leq \left((n+1)e + \frac{8e^{1/5}}{n(2n-1)(2\pi)^{2n}}\right)\left(\frac{n}{x}\right)^{2n}. \tag{1.3}$$

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2 The proof of the theorem

2.1 Explicit formula for the coefficients

The proof of our theorem is based on the following well-known and more precise version of the Stirling-De Moivre series [3].

Theorem 2. *Let $x \geq 1$, then for every $n \geq 1$, the following expression holds:*

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} + S_n(x), \tag{2.1}$$

where the $S_n(x)$ remainder satisfies

$$|S_n(x)| \leq \frac{|B_{2n}|}{2n(2n-1)x^{2n-1}}, \tag{2.2}$$

where B_k denotes the k^{th} Bernoulli number. ∎

Proof. A simple algebraic manipulation of the above expansion gives

$$\frac{1}{x} \log\left(\frac{\Gamma(x) e^x}{x^{x-1/2}\sqrt{2\pi}}\right) = \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k}} + \frac{S_n(x)}{x}. \tag{2.3}$$

Taking the exponential of both sides gives

$$\left(\frac{\Gamma(x) e^x}{x^{x-1/2}\sqrt{2\pi}}\right)^{1/x} = \exp\left(\sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k}}\right) \cdot \exp\left(\frac{S_n(x)}{x}\right). \tag{2.4}$$

By the Maclaurin series of the exponential function we obtain

$$\exp\left(\sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k}}\right) \cdot \exp\left(\frac{S_n(x)}{x}\right) \tag{2.5}$$

$$= \exp\left(\frac{S_n(x)}{x}\right) \cdot \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) \tag{2.6}$$

$$= \exp\left(\frac{S_n(x)}{x}\right) \cdot \prod_{k=1}^{n-1} \sum_{l \geq 0} \frac{1}{l!} \left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)^l. \tag{2.7}$$

Applying the Cauchy product and collecting the coefficients of x^{-2k} we obtain

$$\exp\left(\frac{S_n(x)}{x}\right) \cdot \prod_{k=1}^{n-1} \sum_{l \geq 0} \frac{1}{l!} \left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)^l = \sum_{k=0}^{n-1} \frac{G_k}{x^{2k}} + R_n(x), \tag{2.8}$$

where the G_k coefficients are given by

$$G_k = \sum_{\substack{m_1, m_2, \ldots, m_k \geq 0 \\ 2m_1 + 4m_2 + \ldots + 2km_k = 2k}} \prod_{r=1}^{k} \frac{1}{m_r!} \left(\frac{B_{2r}}{2r(2r-1)}\right)^{m_r}, \quad G_0 = 1, \tag{2.9}$$

and $R_n(x) = \mathcal{O}\left(\frac{1}{x^{2n}}\right)$. This completes the proof of the first part of the theorem. ∎

2.2 The estimate of the remainder

We will estimate uniformly by x and n the value of the remainder $R_n(x)$ by the method of Karatsuba [1].

Proof. In the first part of the proof we saw that

$$\exp\left(\frac{S_n(x)}{x}\right)\cdot\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)=\sum_{k=0}^{n-1}\frac{G_k}{x^{2k}}+R_n(x). \tag{2.10}$$

First let

$$\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)=\sum_{j\geq 0}\frac{K_j}{x^j}, \tag{2.11}$$

thus $K_{2j}=G_j$ if $0\leq j\leq n-1$ and $K_{2j+1}=0$ if $j\geq 0$. Let us verify the order of the factor $\exp\left(\frac{S_n(x)}{x}\right)$. Assume that $x\geq n+1$, then

$$\left|\frac{S_n(x)}{x}\right|\leq\frac{|B_{2n}|}{2n(2n-1)x^{2n}}\leq\frac{4}{n(2n-1)x^{2n}}\left(\frac{n}{2\pi}\right)^{2n}<\frac{1}{9} \tag{2.12}$$

by the well known inequality

$$|B_{2n}|\leq 8\left(\frac{n}{2\pi}\right)^{2n}. \tag{2.13}$$

On the other hand we have

$$\exp\left(\frac{S_n(x)}{x}\right)=\sum_{k\geq 0}\frac{1}{k!}\left(\frac{S_n(x)}{x}\right)^k=1+2\delta_n\frac{S_n(x)}{x},\ 0<\delta_n<1. \tag{2.14}$$

Now by the properties of the Stirling series one can find

$$\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}+\frac{S_n(x)}{x}\right)<\exp\left(\frac{1}{12x}\right)\leq\exp\left(\frac{1}{12}\right). \tag{2.15}$$

From (2.12) and (2.15) we find that

$$\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)=\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}+\frac{S_n(x)}{x}-\frac{S_n(x)}{x}\right)<\exp\left(\frac{1}{5}\right). \tag{2.16}$$

From (2.14) and (2.16) we have

$$\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}+\frac{S_n(x)}{x}\right)=\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)\left(1+2\delta_n\frac{S_n(x)}{x}\right) \tag{2.17}$$

$$=\prod_{k=1}^{n-1}\exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)+2\eta_n e^{1/5}\frac{S_n(x)}{x} \tag{2.18}$$

for $-1\leq\eta_n\leq 1$. Comparing (2.10), (2.11) and (2.18) we find the following expression for our remainder

$$R_n(x)=\sum_{j\geq 2n-1}\frac{K_j}{x^j}+2\eta_n e^{1/5}\frac{S_n(x)}{x},\ x\geq n+1 \tag{2.19}$$

and

$$|R_n(x)| \le \left|\sum_{j\ge 2n-1} \frac{K_j}{x^j}\right| + 2e^{1/5}\left|\frac{S_n(x)}{x}\right| \le \sum_{j\ge 2n-1} \frac{|K_j|}{x^j} + \frac{e^{1/5}|B_{2n}|}{n(2n-1)x^{2n}}. \tag{2.20}$$

We are going to estimate the value of the sum. Let

$$f(z) := \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)} z^{2k}\right) = \sum_{j\ge 0} K_j z^j. \tag{2.21}$$

Hence the K_j coefficients are the coefficients in the Taylor expansion of f. Thus for every $w > 0$ we have

$$K_j = \frac{1}{2\pi i}\int_{|v|=w} \frac{f(v)}{v^{j+1}}dv. \tag{2.22}$$

From this

$$|K_j| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{w\left|f\left(we^{i\varphi}\right)\right| d\varphi}{w^{j+1}} \le \max_{0\le\varphi\le 2\pi} \frac{\left|f\left(we^{i\varphi}\right)\right|}{w^j}. \tag{2.23}$$

Moreover,

$$\left|f\left(we^{i\varphi}\right)\right| \le \prod_{k=1}^{n-1} \exp\left(\frac{|B_{2k}|}{2k(2k-1)} w^{2k}\right) = \exp\left(\sum_{k=1}^{n-1} \frac{|B_{2k}|}{2k(2k-1)} w^{2k}\right). \tag{2.24}$$

Let $w = 1/n$, from (2.13)

$$\sum_{k=1}^{n-1} \frac{|B_{2k}|}{2k(2k-1)} w^{2k} \le \sum_{k=1}^{n-1} \frac{4}{k(2k-1)}\left(\frac{kw}{2\pi}\right)^{2k} \le \sum_{k=1}^{n-1} \frac{4}{k(2k-1)}\left(\frac{1}{2\pi}\right)^{2k} < 1. \tag{2.25}$$

Hence $\left|f\left(we^{i\varphi}\right)\right| \le e$ valid when $w = 1/n$ and by (2.23) we finally have

$$|K_j| \le en^j. \tag{2.26}$$

Substituting this into (2.20) and noting that $K_{2n-1} = 0$ we obtain

$$|R_n(x)| \le \sum_{j\ge 2n-1} \frac{en^j}{x^j} + \frac{e^{1/5}|B_{2n}|}{n(2n-1)x^{2n}} = e\left(\frac{n}{x}\right)^{2n} \frac{1}{1-n/x} + \frac{e^{1/5}|B_{2n}|}{n(2n-1)x^{2n}} \tag{2.27}$$

$$\le (n+1)e\left(\frac{n}{x}\right)^{2n} + \frac{8e^{1/5}}{n(2n-1)x^{2n}}\left(\frac{n}{2\pi}\right)^{2n} \tag{2.28}$$

$$= \left((n+1)e + \frac{8e^{1/5}}{n(2n-1)(2\pi)^{2n}}\right)\left(\frac{n}{x}\right)^{2n}, \tag{2.29}$$

which completes the proof. ■

3 Numerical properties

3.1 The computation of the coefficients

From (1.2) we can compute the numerical values of G_k. The Bernoulli numbers start

$$B_j = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, \ldots (j \ge 0).$$

So the first few values of the G_k coefficients are

$$\begin{aligned}
G_0 &= 1 \\
G_1 &= \frac{1}{2}B_2 = \frac{1}{12} \\
G_2 &= \frac{1}{12}B_4 + \frac{1}{8}B_2^2 = \frac{1}{1440} \\
G_3 &= \frac{1}{30}B_6 + \frac{1}{24}B_2B_4 + \frac{1}{48}B_2^3 = \frac{239}{362880} \\
G_4 &= \frac{1}{56}B_8 + \frac{1}{60}B_2B_6 + \frac{1}{288}B_4^2 + \frac{1}{96}B_4B_2^2 + \frac{1}{384}B_2^4 = -\frac{46409}{87091200}
\end{aligned}$$

Thus the expansion (1.1) starts

$$\Gamma(x) \sim \left(\frac{x}{e}\left(1 + \frac{1}{12x^2} + \frac{1}{1440x^4} + \frac{239}{362880x^6} - \frac{46409}{87091200x^8} + \ldots\right)\right)^x \sqrt{\frac{2\pi}{x}}. \tag{3.1}$$

The following table shows the numerical values of the first eleven coefficients.

G_0	1.00000000000000000000000000000000000
G_1	0.08333333333333333333333333333333333
G_2	0.00069444444444444444444444444444444
G_3	0.00065861992945326278659611992945326
G_4	-0.00053287817827748383303938859494415
G_5	0.00079278588700608376534302460228386
G_6	-0.00184758189322033028400606295961969
G_7	0.00625067824784941846328836824623616
G_8	-0.02901710246301150993444701506844402
G_9	0.17718457242491308890302832366796470
G_{10}	-1.37747681703993534399676348903067470

Table with numerical values of G_k.

If we formally write

$$\exp\left(\sum_{k\geq 1} \frac{B_{2k}}{2k\,(2k-1)\,x^{2k}}\right) = \sum_{k\geq 0} \frac{G_k}{x^{2k}}, \tag{3.2}$$

then the following recurrence holds for $k \geq 1$.

$$G_k = \frac{1}{2k} \sum_{m=0}^{k-1} \frac{B_{2m+2}G_{k-1-m}}{2m+1}, \quad G_0 = 1, \tag{3.3}$$

which gives the same as above. This can be shown by differentiating both sides of (3.2) and equating the coefficients of x^{-2k}.

3.2 Numerical comparisons

Although we considered an asymptotic formula, i. e. a formula which is optimized for use with *large* values of x, it is for practical purposes also of interest to know the behaviour for *small* values of x. Therefore we will compare in this paragraph the numerical performance of some asymptotic formula to the Gamma function with our formula in the range $x \in [0.5 \,..\, 50]$. We compare the following approximation formulas [5,6]

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - ...\right) \quad \text{(Stirling)}\,, \tag{3.4}$$

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - ...\right) \quad \text{(Laplace)}\,, \tag{3.5}$$

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \sqrt[6]{\left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} - ...\right)} \quad \text{(Ramanujan)}\,, \tag{3.6}$$

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x^2} + \frac{1}{1440x^4} + \frac{239}{362880x^6} - ...\right)^x \quad \text{(Nemes)}\,. \tag{3.7}$$

The second expression is sometimes incorrectly called Stirling's formula (see [2]).

The following graph shows the relative error of these formulas to the Gamma function. The plotted quantity is $|\ln(a(x)/\Gamma(x))| \approx |1 - a(x)/\Gamma(x)|$. The first computed values are for $x = 0.5$. Thick traces indicate $a(x) > \Gamma(x)$, thin ones $a(x) < \Gamma(x)$. Color codes for various families: blue - Stirling, green - Laplace, brown - Ramanujan, red - Nemes ($a(x)$ denotes the corresponding approximation). The number after a name indicates the power of $(1/x)$ in the last kept term in the expansion. The following curves overlap: Laplace_2 overlaps Stirling_1, Ramanujan_2 overlaps Nemes_2 for large x, Nemes_4 overlaps Nemes_Closed, Nemes_6 overlaps Stirling_5. For Nemes_Closed formula, see Section 4.1.

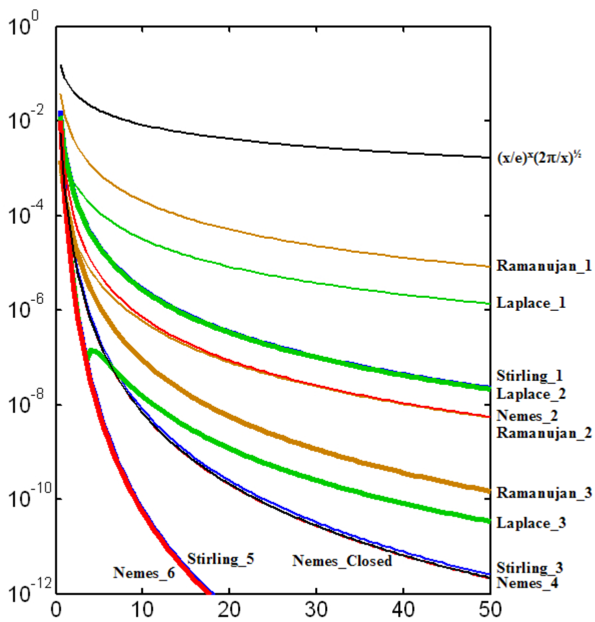

Figure 1: Relative errors of Gamma function approximations.

Conclusion. From the graph we see that the first and the third Laplace approximations outperform the corresponding Ramanujan formulas, however these expansions contain the same number of terms. Ramanujan_2 gives better approximation than Laplace_2 and Nemes_2, though the latter is nearly identical for larger values of x. The graph also shows that, for equal number of 'correction' terms, the Nemes formulae are always better than all the other (e.g., Nemes_4 is better than Stirling_3, Ramanujan_2 and Laplace_2). It seems that for numerical computation the most useful are the Stirling and Nemes formulas. These expansions use only half of the powers of the variable contrary to the Laplace and Ramanujan series. The behavior of the closed formula is very interesting, it gives approximately the same value as Nemes_4 and a better one than Stirling_3, even though it contains only one 'correction' term.

4 Corollaries

4.1 Closed approximation

The structure of the expansion (1.1) induces the following closed approximation to the Gamma function.

Corollary 1. *Let $x \geq 1$, then*

$$\Gamma(x) \sim \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{15x^2}\right)^{\frac{5}{4}x}. \tag{4.1}$$

Proof. If $|t| < 1$

$$(1+t)^{\frac{5}{4}} = 1 + \frac{5}{4}t + \frac{5}{32}t^2 - \frac{5}{128}t^3 + \dots. \tag{4.2}$$

Let $t = \dfrac{1}{15x^2} < 1$, then

$$\left(1 + \frac{1}{15x^2}\right)^{\frac{5}{4}} = 1 + \frac{1}{12x^2} + \frac{1}{1440x^4} - \frac{1}{86400x^6} - \dots, \tag{4.3}$$

which from the approximation is reasonable (compare it with (3.1)). ■

4.2 Asymptotic expansion of $\sqrt[n]{n!}$

From Stirling's formula it is well known that

$$\sqrt[n]{n!} \sim \frac{n}{e}. \tag{4.4}$$

By our new formula we can easily deduce a complete asymptotic expansion for $\sqrt[n]{n!}$.

Corollary 2. *Let n be a positive integer, then*

$$\sqrt[n]{n!} \sim \frac{n}{e} \sum_{k \geq 0} \frac{P_k(\log(2\pi n))}{n^k} \tag{4.5}$$

where P_k is a polynomial in degree k and for every real number x we have

$$P_k(x) = \sum_{j=0}^{k} \frac{V_{k-j}}{2^j j!} x^j, \tag{4.6}$$

where $V_{2i+1} = 0$ and $V_{2i} = G_i$ for $i \geq 0$.

Proof. For positive integer n we have $n\Gamma(n) = n!$, thus by our formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(\sum_{k\geq 0} \frac{G_k}{n^{2k}}\right)^n \tag{4.7}$$

or

$$\sqrt[n]{n!} \sim \frac{n}{e} \sqrt[2n]{2\pi n} \sum_{k\geq 0} \frac{G_k}{n^{2k}}. \tag{4.8}$$

We know that

$$\sqrt[2n]{t} = \exp\left(\frac{1}{2n}\log t\right) = \sum_{j\geq 0} \frac{1}{j!}\left(\frac{\log t}{2n}\right)^j, \tag{4.9}$$

hence setting $t = 2\pi n$ and applying the definition of V_i gives

$$\sqrt[n]{n!} \sim \frac{n}{e} \sum_{j\geq 0} \frac{1}{j!}\left(\frac{\log(2\pi n)}{2n}\right)^j \sum_{k\geq 0} \frac{V_k}{n^k}. \tag{4.10}$$

Now the Cauchy product of the two series gives the desired form. ■

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