New asymptotic expansion for the $\Gamma(z)$ function.

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September 24, 2007

Published in Stan's Library, Volume II, 31 Dec 2007. Link:
www.ebyte.it/library/docs/math07/GammaApproximation.html

Abstract

Using a functional substitution and Faà di Bruno formula, Stirling asymptotic series approximation to the Gamma function is converted into a new one with better convergence properties. The new formula is compared with those of Stirling, Laplace and Ramanujan for real arguments greater than 0.5 and turns out to be, for equal number of 'correction' terms, numerically superior to all of them. As a side benefit, a closed-form approximation has turned up during the analysis which is about as good as 3rd order Stirling's (maximum relative error smaller than 1e-10 for real arguments greater or equal to 10.)

1. The Bell polynomials. Suppose that $h(z) = f(g(z))$ and let

$$h_n = \frac{d^n h(z)}{dz^n},\ f_n = \frac{d^n f(z)}{dz^n},\ g_n = \frac{d^n g(z)}{dz^n};$$

then by applying the chain rule the first few derivatives of $h(z)$ are given by

$$h_1 = f_1 g_1,$$

$$h_2 = f_1 g_2 + f_2 g_1^2,$$

$$h_3 = f_1 g_3 + f_2 3 g_1 g_2 + f_3 g_1^3.$$

The general form is given by Faà di Bruno's formula

$$h_n = \sum_{k=1}^{n} f_k \mathrm{B}_{n,k}(g_1, g_2, ..., g_{n-k+1}).$$

Here $\mathrm{B}_{n,k}$ denotes a homogeneous polynomial of degree k and weight n in the g_m, called the Bell polynomial (see [2]). Another notation is

$$\mathrm{B}_n(g_1, g_2, ..., g_n) = \sum_{k=1}^{n} \mathrm{B}_{n,k}(g_1, g_2, ..., g_{n-k+1}), \tag{1}$$

which is the complete Bell polynomial.

They have the explicit form

$$\mathrm{B}_{n,k}(g_1, ..., g_{n-k+1}) = \sum \frac{n!}{j_1!...j_{n-k+1}!}\left(\frac{g_1}{1!}\right)^{j_1} ... \left(\frac{g_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \tag{2}$$

here the sum runs over the set of all partitions of n with k parts, or in other words, over all solutions in non-negative integers j_m of the equations

$$\sum_{m=1}^{n-k+1} j_m = k,\ \sum_{m=1}^{n-k+1} m j_m = n.$$

Another important formula (We will use it to obtain the general term of an asymptotic series.) is the generating function of the complete Bell polynomials given by

$$\mathcal{B}(z) = \exp\left(\sum_{m\geq 1} \frac{g_m z^m}{m!}\right) = 1 + \sum_{n\geq 1} \mathrm{B}_n\left(g_1, g_2, ..., g_n\right)\frac{z^n}{n!}. \tag{3}$$

There is also a more general application in the study of power series, for example let

$$f(z) = \sum_{m\geq 0} a_m \frac{z^m}{m!}, \quad g(z) = \sum_{m\geq 0} b_n \frac{z^n}{n!},$$

then

$$f(g(z)) = \sum_{n\geq 0} \frac{\sum_{k=1}^{n} a_k B_{n,k}\left(b_1, b_2, ..., b_{n-k+1}\right) z^n}{n!}.$$

2. Stirling's series. Before introducing the new Gamma function approximation, let us first review the proof of the well-known Stirling series (see [1]).

Theorem 1 *Let $z \geq 1$, then*

$$\ln\left(\Gamma(z)\right) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \sum_{n\geq 1} \frac{B_{2n}}{2n\left(2n-1\right)z^{2n-1}}. \tag{4}$$

The series is divergent for all values of z, but the partial sums can be made an arbitrarily good approximation for large enough z. Here the B_n denote the Bernoulli numbers (see [1]).

Proof. In the proof we will apply the Euler-Maclaurin Summation Formula, the Trigamma function and the Wallis product.

The Trigamma function is defined by

$$\psi_1(z) = \frac{d^2 \ln\left(\Gamma(z)\right)}{dz^2} = \sum_{n\geq 0} \frac{1}{(z+n)^2}.$$

We apply the Euler-Maclaurin Formula to find an asymptotic approximation to the function

$$\int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{z}.$$

We get the expression

$$\frac{1}{z} = \int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{2z^2} + \sum_{k\geq 0} \frac{1}{(z+1+k)^2} - \sum_{n\geq 1} \frac{B_{2n}}{z^{2n+1}}.$$

By plugging the Trigamma function into this formula and solving for it we get

$$\psi_1(z+1) = \frac{1}{z} - \frac{1}{2z^2} + \sum_{n\geq 1} \frac{B_{2n}}{z^{2n+1}}.$$

Repeated integration gives an asymptotic series for the Gamma function which, however, contains an integration constant ω.

$$\ln\left(\Gamma(z+1)\right) = \omega + \left(z + \frac{1}{2}\right)\ln(z) - z + \sum_{n\geq 1} \frac{B_{2n}}{2n\left(2n-1\right)z^{2n-1}} \tag{5}$$

In the limit case, this gives the following asymptotic approximation for $\Gamma(z+1)$.

$$\Gamma(z+1) \approx z^{z+\frac{1}{2}} e^{-z} e^{\omega}, \quad \text{as} \quad z \to \infty.$$

To evaluate the constant we apply the Wallis product for π,

$$\frac{\pi}{2}=\prod_{n\geq 1}\frac{(2n)^2}{(2n-1)(2n+1)}.$$

Some manipulation gives the

$$\sqrt{\frac{\pi}{2}}=\lim_{n\to\infty}\frac{(2n)!!}{\sqrt{2n}\,(2n-1)!!}=\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{\sqrt{2n}\,(2n)!} \tag{6}$$

expression. We also have

$$\Gamma(n+1)=n!\approx n^{n+\frac{1}{2}}e^{-n}e^{\omega},$$

if n is an integer. Inserting it into (6) we arrive at

$$\lim_{n\to\infty}\frac{2^{2n}\left(n^{2n+1}e^{-2n}e^{2\omega}\right)}{(2n)^{2n+\frac{1}{2}}e^{-2n}e^{\omega}}\frac{1}{\sqrt{2n}}=\sqrt{\frac{\pi}{2}}\Leftrightarrow\omega\to\frac{1}{2}\ln(2\pi).$$

Replacing ω in (5) by $\frac{1}{2}\ln(2\pi)$ completes the proof. ∎

3. The new asymptotic series. In this section we introduce a new series approximation to the Gamma function for large values of z.

Theorem 2 *Let $z\geq 1$, then the following asymptotic series is valid*

$$\Gamma(z)=\sqrt{\frac{2\pi}{z}}e^{-z}\left(z+\sum_{n\geq 1}\frac{b_n}{z^{n-1}}\right)^{z}, \tag{7}$$

as $z\to\infty$. The b_n coefficients are given by

$$b_n=\mathrm{B}_n(0,B_2,...,(n-2)!B_n)\frac{1}{n!}. \tag{8}$$

Proof. First we rewrite Stirling's series in the form

$$\frac{1}{z}\ln\left(\Gamma(z)\left(\frac{e}{z}\right)^{z}\sqrt{\frac{z}{2\pi}}\right)=\sum_{n\geq 1}\frac{B_{2n}}{2n(2n-1)z^{2n}}.$$

Taking the exponential of each side gives

$$\frac{e}{z}\sqrt[z]{\Gamma(z)\sqrt{\frac{z}{2\pi}}}=\exp\left(\sum_{n\geq 1}\frac{B_{2n}}{2n(2n-1)z^{2n}}\right). \tag{9}$$

To expand the right side of the equation to a series we can apply the generating function of the complete Bell polynomial (3) that we mentioned above. Here the variable is z^{-1} instead of z. If we use the same notations the g_n coefficients take the form

$$g_1=0\ \text{ and }\ g_n=\frac{n!B_n}{n(n-1)}=(n-2)!B_n,\ \text{ if }\ n\geq 2.$$

Since $B_3=B_5=B_7=...=0$. This means that our expression (9) becomes

$$\frac{e}{z}\sqrt[z]{\Gamma(z)\sqrt{\frac{z}{2\pi}}}=1+\sum_{n\geq 1}\mathrm{B}_n(0,B_2,...,(n-2)!B_n)\frac{1}{n!z^n}.$$

By solving for $\Gamma(z)$ we finally arrive at our asymptotic formula and it completes the proof. ∎

The computation of the coefficients. From (1), (2) and (8) we can compute the numerical values of b_n. The Bernoulli numbers start

$$B_n = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, \ldots (n \geq 0) .$$

So $b_n = 0$ if n is odd and the first few values of the even-indexed coefficients are

$$\begin{aligned} b_2 &= \frac{1}{2} B_2 = \frac{1}{12} \\ b_4 &= \frac{1}{12} B_4 + \frac{1}{8} B_2^2 = \frac{1}{1440} \\ b_6 &= \frac{1}{30} B_6 + \frac{1}{24} B_2 B_4 + \frac{1}{48} B_2^3 = \frac{239}{362880} \\ b_8 &= \frac{1}{56} B_8 + \frac{1}{60} B_2 B_6 + \frac{1}{288} B_4^2 + \frac{1}{96} B_4 B_2^2 + \frac{1}{384} B_2^4 = -\frac{46409}{87091200} \end{aligned}$$

Thus the expansion (7) starts

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} e^{-z} \left(z + \frac{1}{12z} + \frac{1}{1440z^3} + \frac{239}{362880z^5} - \frac{46409}{87091200z^7} + \ldots \right)^z .$$

b_2	0.08333333333333333333333333333333333
b_4	0.00069444444444444444444444444444444
b_6	0.00065861992945326278659611992945326
b_8	-0.00053287817827748383303938859494415
b_{10}	0.00079278588700608376534302460228386
b_{12}	-0.00184758189322033028400606295961969
b_{14}	0.00625067824784941846328836824623616
b_{16}	-0.02901710246301150993444701506844402
b_{18}	0.17718457242491308890302832366796470
b_{20}	-1.37747681703993534399676348903067470

Table with numerical values of b_n.

Peter Luschny showed that this asymptotic series can be expanded into a half-integer continued fraction formula

$$n! = \sqrt{2\pi} \left(n + \frac{1}{2} \right)^{\left(n+\frac{1}{2}\right)} e^{-\left(n+\frac{1}{2}\right)} \left(\cfrac{\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right) + \cfrac{\frac{1}{24}}{\left(n+\frac{1}{2}\right) + \cfrac{\frac{3}{80}}{\ddots}}} \right)^{\left(n+\frac{1}{2}\right)} . \qquad (10)$$

This formula is particularly useful if we want to approximate the factorial function for both small and large values of n (see [4]).

4. Numerical comparisons. Although we considered an asymptotical formula, i. e. a formula which is optimized for use with *large* values of x, it is for practical purposes also of interest to know the behaviour for *small* values of x. Therefore we will compare in this paragraph the numerical performance of some asymptotic formula to the Gamma function with our formula in the range $x \in [0.5 \,..\, 50]$.

To this aim we introduce the relative error of an approximation $a(x)$ to $\Gamma(x)$. This can be defined as

$$\delta(x, a) = 1 - \frac{a(x)}{\Gamma(x)} . \qquad (11)$$

We compare the following approximation formulas

$$\sigma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - ...\right) \quad \text{(Stirling series)}\,,$$
$$\lambda(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - ...\right) \quad \text{(Laplace formula)}\,,$$
$$\rho(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \sqrt[6]{\left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} - ...\right)} \quad \text{(Ramanujan formula)}\,,$$
$$\nu(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x^2} + \frac{1}{1440x^4} + \frac{239}{362880x^6} - ...\right)^x \quad \text{(Nemes formula)}\,.$$

The second expression is sometimes incorrectly called Stirling's formula (see [6]).

The following graph showes the relative error of these formulas to the Gamma function. The plotted quantity is $|\ln(a(x)/\Gamma(x))| \approx |1 - a(x)/\Gamma(x)|$. The first computed values are for $x = 0.5$. Thick traces indicate $a(x) > \Gamma(x)$, thin ones $a(x) < \Gamma(x)$. Color codes for various families: blue - Stirling, green - Laplace, brown - Ramanujan, red - Nemes. The number after a name indicates the power of $(1/x)$ in the last kept term in the expansion. The following curves overlap: Laplace_2 overlaps Stirling_1, Ramanujan_2 overlaps Nemes_2 for large x, Nemes_4 overlaps Nemes_Closed, Nemes_6 overlaps Stirling_5. For Nemes_Closed formula, see the Section 5.

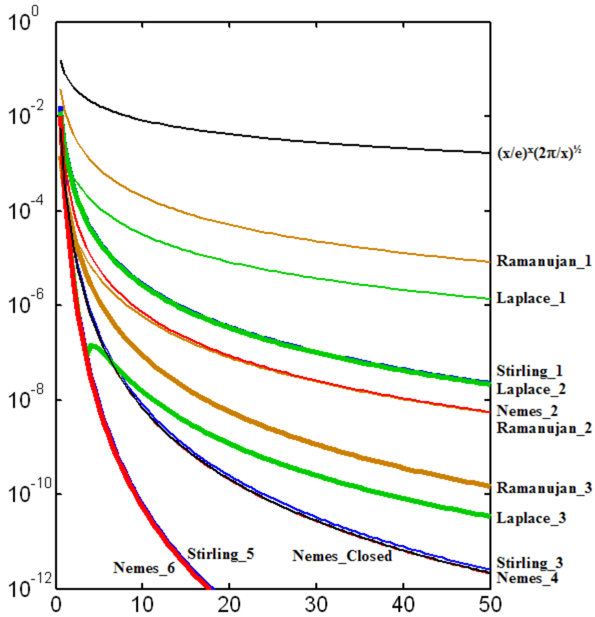

Figure 1: Relative errors of Gamma function approximations.

Conclusion. From the graph we see that the first and the third Laplace approximations outperform the corresponding Ramanujan formulas, but these expansions do not contain the same number of terms. Ramanujan_2 gives better approximation than Laplace_2 and Nemes_2, though the latter is nearly identical for larger values of x. The graph also shows that, for equal number of 'correction' terms, the Nemes formulae are always better than all the other (e.g., Nemes_4 is better than Stirling_3, Ramanujan_2 and Laplace_2). The Stirling and Nemes formulas are very useful, because they use only half of the powers of the variable contrary to the Laplace and Ramanujan series.

The behavior of the closed formula is very interesting, it gives approximately the same value as Nemes_4 and a better one than Stirling_3, even though it contains only one 'correction' term.

5. An interesting closed approximation. We can get a good closed approximation to the Gamma function if we approximate the Laplace series by a much simpler function. Write up Laplace's formula in the form

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \frac{163879}{209018880z^5} + \ldots\right).$$

There exists a function which has a very similar series expansion

$$\left(1 + \frac{1}{15z^2}\right)^{\frac{5}{4}z} = 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \frac{16997}{149299200z^5} + \ldots .$$

By exchanging the series for this function we get

$$\Gamma(z) \approx \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}} \left(1 + \frac{1}{15z^2}\right)^{\frac{5}{4}z} \tag{12}$$

which is valid for large values of z. This is Nemes_Closed in the comparison paragraph.

Acknowledgement. I want to thank Peter Luschny, Péter Simon and Stanislav Sýkora for discussing with me some of the points in this paper.

Comments. Peter Luschny gave another definition of the b_n coefficients which is equivalent to the formula we obtained above. If we write our formula in the form

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e} \sum_{n \geq 0} \frac{b_{2n}}{z^{2n}}\right)^z, \tag{13}$$

the b_{2n} coefficients can be computed by

$$\sum_{\sum_1^j p_i \mu_i = 2n} \frac{B_{p_1}^{\mu_1}}{\mu_1! \left(p_1^2 - p_1\right)^{\mu_1}} \frac{B_{p_2}^{\mu_2}}{\mu_2! \left(p_2^2 - p_2\right)^{\mu_2}} \cdots \frac{B_{p_j}^{\mu_j}}{\mu_j! \left(p_j^2 - p_j\right)^{\mu_j}}. \tag{14}$$

Here the sum runs over all partitions of $2n$ with even parts.

The error curves were generated by Stanislav Sýkora using Matlab. For more information about this please visit his website: www.ebyte.it

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