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Sequences related to the differential equation f'' = af'f

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This Note explores a family of sequences related through their exponential generating functions (e.g.f.) to smooth solutions of the nonlinear differential equation f''(z) = af'(z)f(z) in *C*. The implied recurrence for the complex expansion coefficients of the solutions leads to a family of sequences uniquely labeled by two complex parameters. Excluding a subset of 'singular' cases, each of these sequences has an e.g.f. which can be written as a multiple of the tangent function of a linear form of its complex argument. There exists also a closely related family of singular cases which lead instead to a simple inverse linear form.

A subset of this family contains sequences with exclusively integer elements, one for every pair of integer numbers. Since many of these integer sequences appear on OEIS, often without an explicit recognition of the close relationship between them, it is hoped that the present Note might constitute a useful unifying link and a useful classification. Another interesting subset is that of sequences with complex coefficients which have integer real and imaginary parts. These seem to be novel so far, and a couple of examples are listed.

Keywords: mathematics, sequence, e.g.f., generating function, expansion, sequence

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I. Introduction

We will consider functions $f(z)$ in the complex [1] domain $\boldsymbol{C}$, which satisfy the nonlinear differential equation

$$y''(z) = a.y'(z)y(z), \tag{I.1}$$

with a being some non-zero complex constant. Notice, however, that setting

$$y(z) = f(z)/a, \tag{I.2}$$

the equation simplifies to

$$f''(z) = f'(z)f(z). \tag{I.3}$$

It is therefore possible, without any loss of generality, to drop the constant a and focus just on equation (I.3).

We will consider only those 'solutions' $f(z)$ of equation (I.3) which are analytic [2] in a neighborhood of $z = 0$, i.e., have therein a convergent expansion in powers of z:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \tag{I.4}$$

with f_k being some constant complex coefficients (here, we will indicate the set of such functions as F). Then

$$f'(z) = \sum_{n=0}^{\infty}(n+1)f_{n+1}\, z^n \tag{I.5}$$

$$f''(z) = \sum_{n=0}^{\infty}(n+1)(n+2)f_{n+2}\, z^n \tag{I.6}$$

and the condition (I.3) implies the recurrence

$$(n+1)(n+2)f_{n+2} = \sum_{k=0}^{n}(n-k+1)f_k f_{n-k+1} \tag{I.7}$$

An elementary manipulation shows that setting

$$f_n = c_n/n!\,, \tag{I.8}$$

the recurrence becomes

$$c_{n+2} = \sum_{k=0}^{n} C(n,k) c_k c_{n-k+1} \tag{I.9}$$

where $C(n,k)$ are the binomial coefficients [3]. Explicitly, the first few terms of the recurrence are:

$$c_2 = c_0c_1, c_3 = c_0c_2 + c_1^2, c_4 = c_0c_3 + 3c_1c_2,\ \ c_5 = c_0c_4 + 4c_1c_3 + 3c_2^2,\ \ c_6 = c_0c_5 + 5c_1c_4 + 10c_2c_3,\ \ldots \tag{I.9a}$$

Substituting expression (I.8) into the equation (I.4), we realize that $f(z)$ is the *exponential generating function* [4] (e.g.f.) of the sequence $\{c_n\}$,

$$f(z) = \sum_{n=0}^{\infty} (c_n/n!) z^n\,, \tag{I.10}$$

and the c_n are therefore the complex coefficients of the Taylor-Maclaurin series [5] for $f(z)$.

To define a specific instance of the recurrence (I.9) one needs to select just the first two terms c_0 and c_1. The recurrence therefore gives rise to a family of sequences (in general complex-valued) with two settable parameters. We will denote the members of this family of sequences as $S\{c_0,c_1\}$ and the corresponding e.g.f. as $egf(z;S\{c_0,c_1\})$. In addition, we will also denote the n-th element of $S\{c_0,c_1\}$ as $c_n \equiv S_n\{c_0,c_1\}$.

From the above it follows that the following uniqueness lemma holds:

Lemma 1: If two functions $f(z)$ and $g(z)$ both belong to F and $f(0) = g(0)$, $f'(0) = g'(0)$, then (a) each of them is the e.g.f. of a unique sequence, and (b) if the two sequences are identical, $f(z) \equiv g(z)$.

Proof: Point (a) is a direct consequence of the above explicit construction of the sequence having $f(z)$ as its e.g.f. Point (b) follows from the fact that the imposed conditions define, in virtue of (I.10), the two starting values of the two sequences, namely $c_0 = f(0) = g(0)$ and $c_1 = f'(0) = g'(0)$. Since the starting values of the recurrence process are the same, the two sequences are identical [6], implying also the identity of the two functions.

II. Basic properties of the sequences

Lemma 2: When $c_1 = 0$ then $S_n\{c_0,c_1\} = 0$ for any $n > 0$, and therefore $egf(z;S\{c_0,0\}) = c_0 = constant$.

Proof by induction: From (I.9) we have $c_2 = c_0c_1, c_3 = c_0c_2 + c_1^2, c_4 = c_0c_3 + 3c_1c_2, \ldots$, illustrating that, for any $n \geq 2$, the coefficient c_n can be written as a sum of terms, the first one of which is a product of the possibly nonzero c_0 with c_{n-1}, while the others contain as factors coefficients with indices larger than 0, but smaller than n. Hence, if the Lemma holds any nonzero index smaller than n, then it holds also for c_n.

Since constant functions are trivial solutions of (I.1) and (I.3), we will henceforth assume that $c_1 \neq 0$.

Lemma 3: When $c_0 = 0$ then, by equations (I.9), $S_n\{c_0,c_1\} = 0$ for all even-indices, and therefore $egf(z;S\{0,c_1\})$ is an odd function.

Proof by induction: When n is even then each term on the right-hand side of (I.9) contains the product of two coefficient $c_k c_{n-k+1}$ one of which has an even index. Therefore, assuming that the Lemma holds for every even index up to some n, then it holds also for $n+2$.

Lemma 4: Given any constant a, when (I.3) holds for a function $f(z)$, then it holds also for $a \cdot f(az)$.

Proof: Use direct substitution and the elementary properties of the derivatives.

In terms of the corresponding sequence, this implies

Lemma 5: Given any sequence $S\{c_0, c_1\}$ and any complex constant a, we have

$$egf(z; S\{ac_0, a^2c_1\}) = a * egf(az; S\{c_0, c_1\}) \tag{II.1}$$

and, for every $n = 0, 1, 2, \ldots$,

$$S_n\{ac_0, a^2c_1\} = a^{n+1} S_n\{c_0, c_1\}. \tag{II.2}$$

Proof: Set $f(z) = egf(z; S\{ac_0, a^2c_1\})$ and $g(z) = a * egf(az; S\{c_0, c_1\})$. Since these two functions satisfy the pre-requisites of Lemma 1 (they have the same value and the same first derivatives at $z = 0$), they must be identical. The last identity is easily verified by comparing the corresponding terms in the expansion (I.10) for $f(z)$ and $g(z)$, which must be identical as well.

An important special case of the last Lemma arises when $a = -1$, corresponding to a sign inversion of c_0, while c_0 does not change:

Lemma 6: $egf(z; S\{-c_0, c_1\}) = -egf(-z; S\{c_0, c_1\})$, and therefore the elements of $S\{-c_0, c_1\}$ differ from those of $S\{c_0, c_1\}$ only by the alternation of the signs of even terms,

$$S_n\{-c_0, c_1\} = -(-1)^n S_n\{c_0, c_1\}. \tag{II.3}$$

When c_0, c_1 are integer numbers then, in virtue to the recurrence (I.9), all the terms $S_n\{c_0, c_1\}$ are integer as well. It is likewise obvious that when c_0, c_1 are rational numbers, then all the terms of $S_n\{c_0, c_1\}$ are also rational. Moreover:

Lemma 7: When c_0, c_1 are rational numbers and q is the smallest common multiple of their denominators, then

$$egf(q * z; S\{c_0, c_1\}) = egf(z/q\,; S\{qc_0, q^2c_1\})/q, \tag{II.4}$$

and $S\{qc_0, q^2c_1\}$ is an integer sequence.

Proof: In Lemma 5, set $a = q$ and $z \to z/q$.

In this way, the e.g.f. of any sequence $S\{c_0, c_1\}$ generated by a pair of rational numbers can be expressed by means of the e.g.f. of an integer sequence. Other interesting special cases of Lemma 5 arise by setting $a = \pm j$:

Lemma 8: $egf(z; S\{\pm jc_0, -c_1\}) = \pm j \cdot egf(\pm jz; S\{c_0, c_1\})$, and therefore

$$S_n\{\pm jc_0, -c_1\} = (\pm j)^{n+1}\, S_n\{c_0, c_1\}. \tag{II.5}$$

Consequently, when c_0, c_1 are real numbers, a sequence $S\{\pm jc_0, c_1\}$ alternates purely imaginary terms with purely real ones.

It is also interesting that, in general, integer complex coefficients c_0, c_1 give rise to a sequence of integer complex numbers (see Section IX).

III. Closed expressions for the e.g.f.

From equation (I.3) we obtain, after a rearrangement and a first integration,

$$f'' = f'f = (f^2)'/2 \Longrightarrow 2f' = f^2 + \kappa^2, \tag{III.1}$$

with κ being any arbitrary constant. Setting $f(z) = \kappa g(z)$, this gives

$$dg/(g^2+1) = (\kappa/2)dz \Longrightarrow d\big(atan(g)\big) = d(\lambda_0 z), \tag{III.2}$$

where $\lambda_0 = \kappa/2$ is still a constant. Integrating the latter equation, we obtain

$$atan\big(g(z)\big) = \lambda_0 z + \lambda_1 \Longrightarrow g(z) = tan(\lambda_0 z + \lambda_1), \tag{III.3}$$

where $tan(z)$ and $atan(z)$ are the tangent [8, 9, 10] and the inverse[1] tangent functions [11, 12, 13], respectively.
Equation (III.3) yields

$$f(z) = 2\lambda_0 tan(\lambda_0 z + \lambda_1). \tag{III.4}$$

In order to fully define the $egf(z; S\{c_0, c_1\})$, we need to find the integration constants $\{\lambda_0, \lambda_1\}$ matching the two starting coefficients $\{c_0, c_1\}$.
This can be easily done by comparing the derivatives:

$$c_0 = f(0) = 2\lambda_0 tan(\lambda_1), \quad c_1 = f'(0) = 2\lambda_0^2/cos^2(\lambda_1), \tag{III.5}$$

Equations (III.5) represent a mapping from $\{\lambda_0, \lambda_1\}$ to $\{c_0, c_1\}$ which is unique and void of problems, as long as λ_0 is nonzero and λ_1 is different from $\pm\pi/2 + 2k\pi$. The inverse mapping, from $\{c_0, c_1\}$ to $\{\lambda_0, \lambda_1\}$, requires a bit of care. After some manipulation, the result is

$$\lambda_1 = acos\left(\sqrt{(2c_1 - {c_0}^2)/(2c_1)}\right), \;\; \lambda_0 = \sigma * \sqrt{(2c_1 - {c_0}^2)/4}\;, \;\; \sigma = c_0 c_1/\sqrt{c_0^2 c_1^2}. \tag{III.6a}$$

[1] For brevity, we use the notations atan, acos, asine for the inverse functions of tan, cos, sine, respectively, instead of the notations arctan, arccos, arcsine. Likewise, atanh, acosh, asinh are used for the inverse hyperbolic functions.

Using the principal value of the square root function [14] and the main branch [15] of $acos(z)$, the solution (III.6a) covers any complex values of $\{c_0, c_1\}$, except for two special situations. We will see in a moment that one of these situations turns out to be still 'regular', while the other is 'singular' and leads to a different type of e.g.f.

The first special situation occurs when $c_0 = 0$ since, in this case, equations (III.6a) formally fail unless one resorts to a limit transition. However, the following, easily verifiable, solution of for this situation emerges from equations (III.5):

$$c_0 = 0 \quad \Longrightarrow \quad \lambda_1 = 0,\ \ \lambda_0 = \sigma * \sqrt{c_1/2}\ ,\ \ \sigma = c_1/\sqrt{c_1^2}. \tag{III.6b}$$

The other special situation is singular and arises for $c_1 = {c_0}^2/2$. Equations (III.6a) indicate that when $c_1 \longrightarrow {c_0}^2/2$ then $\lambda_0 \longrightarrow 0$, $\lambda_1 \longrightarrow \pi/2$, and therefore $f(z) \rightarrow \infty$ for any z. Moreover, since there are many pairs $\{c_0, c_1\}$ for which $c_1 = {c_0}^2/2$ and they all lead to $\lambda_0 = 0$, $\lambda_1 = \pi/2$. This means that the inverse mapping $\{\lambda_0, \lambda_1\} \longrightarrow \{c_0, c_1\}$ is for these values of $\{\lambda_0, \lambda_1\}$ undefined.

In order to resolve this situation note that, by Lemma 5, $egf(z; S\{c_0, c_0^2/2\}) = c_0 \cdot egf(c_0 z; S\{1, 1/2\})$.
A very simple proof by induction shows that $S\{1, 1/2\}$ evaluates to

$$S_n\{1, 1/2\} = n!/2^n \ \Longrightarrow \ egf(z; S\{1, 1/2\}) = 1/(1 - z/2). \tag{III.7}$$

Consequently, the singular cases are all covered by the following

Lemma 9: For sequences of the form $S\{c_0, c_0^2/2\}$,

$$egf(z; S\{c_0, c_0^2/2\}) = c_0/(1 - c_0 z/2) \ \text{ and } \ S_n\{c_0, c_0^2/2\} = c_0 (c_0/2)^n n! \tag{III.8}$$

It is trivial to verify explicitly that this e.g.f. indeed satisfies equation (I.3).

Having covered all possible sequences $S\{c_0, c_1\}$ we realize that:

Lemma 10: Every sequence $S\{c_0, c_1\}$ defined by the recurrence (I.9) and any complex starting values c_0, c_1 has an e.g.f. which is convergent in some neighborhood[2] of $z = 0$. Explicitly: when $c_1 \neq {c_0}^2/2$, then

$$egf(z; S\{c_0, c_1\}) = 2\lambda_0 tan(\lambda_0 z + \lambda_1), \tag{III.9a}$$

with $\{\lambda_0, \lambda_1\}$ defined either by equations (III.6a) or (III.6b), depending whether c_0 is zero or not. Otherwise, when $c_1 = {c_0}^2/2$, then

$$egf(z; S\{c_0, c_1\}) = c_0/(1 - c_0 z/2). \tag{III.9b}$$

[2] A different matter is the value of the radius of convergence of the e.g.f., which is not investigated here.

To conclude this Section, let us note that when $\{c_0, c_1\}$ are both integer or rational numbers, the argument of the $acos$ function in (III.6a) is the square root of a rational number. Notice also that, since the values under the square roots need not be positive, any of the values of $\{\lambda_0, \lambda_1\}$ may be complex. In such cases, some readers might prefer to switch in equation (III.4 and III.6a) from trigonometric functions and their inverses with complex argument to the corresponding hyperbolic functions [16, 17, 18, 19] and their inverses [20, 22]. We will do that in Section V, but for the moment we will stick to the current notation, privileging the uniformity of exposition.

IV. A few examples

Example 1: Consider the sequence $S\{1,1\}$. Using the PARI/GP software listed in the Appendix, we obtain

$$S\{1,1\} = \{1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, \dots\}, \quad \text{(IV.1a)}$$

which are the Euler zig-zag numbers, alias the alternating permutations counts, OEIS A000111 [27], for which the corresponding e.g.f. is known to be $(1+\sin(z))/\cos(z)$,

It is easy to verify explicitly that, as implied by Lemma 1, this function satisfies equation (I.3). Equations (III.6a) give $\lambda_0 = 0.5$, $\lambda_1 = \pi/4$, and thus, by equation (III.9a),

$$egf(z; S\{1,1\}) = tan(z/2 + \pi/4), \quad \text{(IV.1b)}$$

which is indeed identical to the previous expression.

Example 2: Inverting the sign of c_0 in the previous example inverts the signs of all even terms:

$$S\{-1,1\} = \{-1, 1, -1, 2, -5, 16, -61, 272, -1385, 7936, -50521, \dots\}, \quad \text{(IV.2a)}$$

a sequence for which, according to Lemma 6,

$$egf(z; S\{-1,1\}) = -tan(-z/2 + \pi/4) = tan(z/2 - \pi/4) \equiv (\sin(z) - 1)/\cos(z). \quad \text{(IV.2b}$$

Example 3: Changing the sign of $c_1$in Example 1 does not have such a simple effect as changing the sign of c_0. One obtains:

$$S\{1,-1\} = \{1, -1, -1, 0, 3, 6, -9, -90, -153, 1134, \ 8019, \ 2430, \dots\}. \quad \text{(IV.3a)}$$

After a few elementary simplifications (see Section V below), equations (III.6a) yield for this sequence

$$egf(z; S\{1,-1\}) = \sqrt{3}\, tanh\left(-z\sqrt{3}/2 + acosh\left(\sqrt{3/2}\right)\right). \quad \text{(IV.3b)}$$

Example 4: Let us now see an instance of a sequence with $c_0 = 0$, namely

$$S\{0,2\} = \{0, 2, 0, 4, 0, 32, 0, 544, 0, 15872, 0, \dots\}. \quad \text{(IV.4a)}$$

In this case, equations (III.6b) give $\lambda_0 = 1, \lambda_1 = 0$, so that

$$egf(z; S\{0,2\}) = 2 \cdot tan(z). \quad \text{(IV.4b)}$$

As expected from Lemma 3, all even terms of this sequence are zero and the e.g.f. is odd.
The nonzero odd terms match those of OEIS A012509, a fact that will be discussed in more detail in Section VIII.

Example 5: When, in the previous example, we change the sign of c_1, the result is

$$S\{0,-2\} = \{0, -2, 0, 4, 0, -32, 0, 544, 0, -15872, 0, \dots\}. \quad \text{(IV.5a)}$$

Equations (III.6b) now give $\lambda_0 = j, \lambda_1 = 0$, and therefore

$$egf(z; S\{0,-2\}) = 2j \cdot tan(jz) = -2 \cdot tanh(z). \quad \text{(IV.5b)}$$

This can be generalized as follows:

Lemma 11: When $c_0 = 0$, changing the sign of c_1 results in changing the signs of the alternate odd terms, while all even terms remain zero.

Proof: proceed iteratively, directly from the recursion (I.9).

Notice that the precondition $c_0 = 0$ is essential; when dropped, nothing can be said. To appreciate this, compare Examples 1and 3.

Example 6: Let us also take a look at one of the singular cases with $2c_1 = {c_0}^2$, such as

$$S\{2,2\} = \{2, 2, 4, 12, 48, 240, 1440, 10080, \dots\}, \text{ for which } c_n = 2 * n! \quad \text{(IV.6a)}$$

Equations (III.8) yield for this case the rather self-evident

$$egf(z; S\{2,2\}) = 2/(1 - z), \quad \text{(IV.6b)}$$

This sequence is related to OEIS A208529 but, again, we will discuss this fact later in Section VIII.

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V. Useful identities and simplifications of the e.g.f.'s

The tangent [10] $tan(z)$belongs to the family of trigonometric functions [8, 9] which, in ***C***, are closely related to the hyperbolic ones [16, 17, 18, 19].

The same holds for the respective inverse functions [11, 12, 13, 15, 20, 21, 22], among which we are primarily interested in the inverse cosine [15] $acos(z)$ appearing in (III.6a), and in its hyperbolic counterpart [22] $acosh(z)$. The properties of all these functions are well known and amply discussed in the references.

What interests us here are specifically just a few 'recipes' for simplifying the e.g.f. expressions in order to avoid, when possible, imaginary and complex values, particularly when the starting coefficients $\{c_0, c_1\}$ are real. To do so, if need be, we will privilege the use of the hyperbolic function $tanh(z)$ and $coth(z)$, and of the inverse hyperbolic function $acosh(z)$.

The following identities are valid for any complex z. They are easy to derive from basic definitions and conventions:

$$tan(-z) = -tan(z),\ \ tanh(-z) = -tanh(z) \tag{V.1}$$

$$tan(jz) = j \cdot tanh(z) \tag{V.2}$$

$$tan(z + \pi/2) = -1/tan(z) \equiv -cot(z) \tag{V.3}$$

$$tan(jz + \pi/2) = j \cdot coth(z) \equiv j/\tanh(z) \tag{V.4}$$

$$acos(z) = \left(\sqrt{1-z}/\sqrt{z-1}\right) \cdot acosh(z), \tag{V.5}$$

$$acos\left(\sqrt{-z}\right) = (\pi/2) - asin\left(\sqrt{1+z}\right) = (\pi/2) + \left(\sqrt{-z}/\sqrt{z}\right) \cdot acosh\left(\sqrt{1+z}\right) \tag{V.6}$$

The last two identities are conditioned by the type of complex plane cuts adopted for the multi-valued inverse functions[3]. In this Note we follow the conventions of Abramowitz-Stegun [23], which match those of MathWorld [24] and Mathematica [25], and are respected also by PARI/GP [26]. Occasionally, we will also make use of special values of the involved functions, such as

$$acos(1) = \mathrm{acosh}(1) = 0,\ \ acos\left(\sqrt{1/2}\right) = \pi/4\,,\ \ acos\left(\sqrt{3}/2\right) = \pi/6, \text{etc.} \tag{V.7}$$

[3] Note: The square roots appearing in (V.5) and (V.6) denote the principal values of the function.
Warning: In complex arithmetic, expressions such as $\sqrt{1-z}/\sqrt{z-1}$ or $\sqrt{-z}/\sqrt{z}$ may not be simplified by using a single square root.

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Classification of the e.g.f.'s and their sequences

The e.g.f. functions defined by the roof expression (III.4), $f(z) = 2\lambda_0 tan(\lambda_0 z + \lambda_1)$, can be now grouped, and possibly simplified[4], according for the various types of the pairs $\{\lambda_0, \lambda_1\}$.

In the following, r and s are assumed to be nonnegative real numbers.

Type A: $\lambda_0 = \pm\sqrt{r},\ \lambda_1 = acos(\sqrt{s}),\ with\ s \le 1$ (V.A)

Examining equations (III.6) we see that this applies to sequences with $0 \le c_0^2 < 2c_1$ and that the sign of λ_0 matches that of c_0.

This case requires almost no simplification; we have: $f(z) = 2\sqrt{r}\ tan\left(z\sqrt{r} \pm acos(\sqrt{s})\right)$.

A still more special case arises when $s = 1$ because then $\lambda_1 = 0$ and $f(z) = 2\sqrt{r}\ tan(z\sqrt{r})$

Type B: $\lambda_0 = \pm\sqrt{-r},\ \lambda_1 = acos(\sqrt{-s})$, (V.B)

This case applies to sequences with $0 < 2c_1 < c_0^2$, and the sign of λ_0 matches that of c_0.

The simplified formula is: $f(z) = -2\sqrt{r} \cdot coth\left(z\sqrt{r} \mp acosh(\sqrt{1+s})\right) = -2\sqrt{r}/tanh\left(z\sqrt{r} \mp acosh(\sqrt{1+s})\right)$.

Type C: $\lambda_0 = \pm\sqrt{-r},\ \lambda_1 = acos(\sqrt{s}),\ with\ s \ge 1$: (V.C)

This applies to sequences with $c_1 < 0$, and the sign of λ_0 is opposite than that of c_0.

The simplified formula is: $f(z) = -2\sqrt{r}\ tanh\left(z\sqrt{r} \pm acosh(\sqrt{s})\right)$.

A still more special case arises when $s = 1$ since then $\lambda_1 = 0,\ \lambda_0 = -\sqrt{-r},\ r = -c_1/2$, and $f(z) = 2\sqrt{r}\ tanh(-z\sqrt{r})$.

Type S (singular): corresponds to situations covered by Lemma 9, when $c_1 = {c_0}^2/2$ and $f(z) = c_0/(1 - c_0 z/2)$. (V.S)

[4] To the extent to which equation (V.2), for example, can be considered a 'simplification'. Simplicity is in the eyes of the beholder, so to say.

VI. Table I. Some integer sequences satisfying recurrence (I.9) and their e.g.f.'s

The list is sorted primarily by the absolute value of c_1 (increasing from 1 to 3). For each $|c_1|$ value, there are two rows, one for positive c_1 and one for negative c_1 (marked by gray background). A secondary sorting is according to decreasing c_0 value (from 3 to -1). The first column shows the starting terms of the sequence. The next two columns show values of λ_0 and λ_1 in the $egf(z; S\{c_0, c_1\}) = 2\lambda_0 tan(\lambda_0 z + \lambda_1)$. For uniformity, these are shown in the format used in equations (III.6) even when simpler-looking reduced forms are available. The background color of these cells indicates whether the value is real (white), pure imaginary (bluish), or complex (gray). The penultimate column shows the corresponding e.g.f.'s in their simplified forms (see Section V, which contains also the definition of the distict Types of the e.g.f.'s). Finally, the last column contains notes, such as links to relevant OEIS entries, when they exist.

$S\{c_0, c_1\}$	λ_0	λ_1	Type	$f(z) = egf(z; S\{c_0, c_1\})$	OEIS
3, **1**, 3, 10, 39, 184, 1047, 7000, 53571, 460936, 4404603, 46296040, 530878719, 6595091944, ...	$\sqrt{-7/4}$	$acos\left(\sqrt{-7/2}\right)$	B	$-\sqrt{7}\, coth\left(z\sqrt{7}/2 - acosh(3/\sqrt{2})\right)$	Not in OEIS
3, **-1**, -3, -8, -15, 14, 357, 2302, 7725, -23626, -655383, -6082538, -26422935, 192117134, 5645490477, ...	$-\sqrt{-11/4}$	$acos\left(\sqrt{11/2}\right)$	C	$-\sqrt{11}\, tanh\left(z\sqrt{11}/2 - acosh\left(\sqrt{11/2}\right)\right)$	Not in OEIS
2, **1**, 2, 5, 16, 64, 308, 1730, 11104, 80176, 643232, 5676560, 54650176, 569980384, 6401959328, ...	$\sqrt{-2/4}$	$acos\left(\sqrt{-2/2}\right)$	B	$-\sqrt{2}\, coth\left(z\sqrt{2}/2 - acosh(\sqrt{2})\right)$	c_n = A131178(n), for n>0
2, **-1**, -2, -3, 0, 24, 108, 162, -1440, -14256, -54432, 177552, 4432320, 31796064, 10520928, ...	$-\sqrt{-6/4}$	$acos\left(\sqrt{6/2}\right)$	C	$-\sqrt{6}\, tanh\left(z\sqrt{6}/2 - acosh(\sqrt{3})\right)$	Not in OEIS
1, **1**, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, 22368256, 199360981, 1903757312, ...	$\sqrt{1/4}$	$acos\left(\sqrt{1/2}\right)$	A	$tan(z/2 + \pi/4)$	c_n=A000111(n); example 1
1, **-1**, -1, 0, 3, 6, -9, -90, -153, 1134, 8019, 2430, -262197, -1438074, 4421871, 104152230, ...	$-\sqrt{-3/4}$	$acos\left(\sqrt{3/2}\right)$	C	$-\sqrt{3}\, tanh\left(z\sqrt{3}/2 - acosh\left(\sqrt{3/2}\right)\right)$	Not in OEIS
0, **1**, 0, 1, 0, 4, 0, 34, 0, 496, 0, 11056, 0, 349504, 0, 14873104, 0, 819786496, 0, 56814228736, ...	$\sqrt{2/4}$	$acos\left(\sqrt{2/2}\right)$	A	$\sqrt{2}\, tan(z/\sqrt{2})$	c_{2n-1} = A002105(n), n>0
0, **-1**, 0, 1, 0, -4, 0, 34, 0, -496, 0, 11056, 0, -349504, 0, 14873104, 0, -819786496, 0, 56814228736, 0, ...	$-\sqrt{-2/4}$	$acos\left(\sqrt{2/2}\right)$	C	$-\sqrt{2}\, tanh(z/\sqrt{2})$	see A002105
-1, **1**, -1, 2, -5, 16, -61, 272, -1385, 7936, -50521, 353792, -2702765, 22368256, -199360981, ...	$-\sqrt{1/4}$	$acos\left(\sqrt{1/2}\right)$	A	$tan(z/2 - \pi/4)$	$\|c_n\|$=A000111(n); example 2
-1, **-1**, 1, 0, -3, 6, 9, -90, 153, 1134, -8019, 2430, 262197, -1438074, -4421871, 104152230, ...	$\sqrt{-3/4}$	$acos\left(\sqrt{3/2}\right)$	C	$-\sqrt{3}\, tanh\left(z\sqrt{3}/2 + acosh\left(\sqrt{3/2}\right)\right)$	Not in OEIS
3, **2**, 6, 22, 102, 590, 4110, 33430, 310710, 3248510, 37737150, 482225350, 6722336550, ...	$\sqrt{-5/4}$	$acos\left(\sqrt{-5/4}\right)$	B	$-\sqrt{5}\, coth\left(z\sqrt{5}/2 - acosh(3/2)\right)$	1+2*A230008
3, **-2**, -6, -14, -6, 202, 1506, 4594, -29814, -573062, -4098606, 2741026, 487823034, 6657110122, ...	$-\sqrt{-13/4}$	$acos\left(\sqrt{13/4}\right)$	C	$-\sqrt{13}\, tanh\left(z\sqrt{13}/2 - acosh(\sqrt{13}/2)\right)$	Not in OEIS
2, **2**, 4, 12, 48, 240, 1440, 10080, 80640, 725760, 7257600, 79833600, 958003200, 12454041600, ...	Singular: $c_1 = c_0^2/2$	Lemma 9 Example 6	S	$2/(1-z)$	c_n = A208529(n+2) = 2*n!
2, **-2**, -4, -4, 16, 112, 224, -1696, -18176, -50432, 621056, 8669696, 29403136, -540059648, ...	$-\sqrt{-8/4}$	$acos\left(\sqrt{8/4}\right)$	C	$-\sqrt{8}\, tanh\left(z\sqrt{2} - acosh(\sqrt{2})\right)$	2-2* A006673

1, **2**, 2, 6, 18, 78, 378, 2214, 14562, 108702, 897642, 8171766, 81066258, 871695918, 10091490138,	$\sqrt{3/4}$	$acos\left(\sqrt{3/4}\right)$	A	$\sqrt{3}\,tan(z\sqrt{3}/2+\pi/6)$	c_n = 2*A080635(n) $-\delta_{0,n}$
1, **-2**, -2, 2, 14, 10, -170, -670, 2270, 30490, 26950, -1435150, -8513650, 59564650, 1050090550, ...	$-\sqrt{-5/4}$	$acos\left(\sqrt{5/4}\right)$	C	$-\sqrt{5}\,tanh\left(z\sqrt{5}/2-acosh(\sqrt{5}/2)\right)$	Not in OEIS
0, **2**, 0, 4, 0, 32, 0, 544, 0, 15872, 0, 707584, 0, 44736512, 0, 3807514624, 0, 419730685952, ...	$\sqrt{4/4}=1$	$acos\left(\sqrt{4/4}\right)$	A	$2\,tan(z)$	c_{2n-1} = A012509(n), n>0
0, **-2**, 0, 4, 0, -32, 0, 544, 0, -15872, 0, 707584, 0, -44736512, 0, 3807514624, 0, -419730685952, ...	$-\sqrt{-4/4}$	$acos\left(\sqrt{4/4}\right)$	C	$-2\,tanh(z)$	c_{2n-1} = $(-1)^n$ A012509(n), n>0
-1, **2**, -2, 6, -18, 78, -378, 2214, -14562, 108702, -897642, 8171766, -81066258, 871695918, ...	$-\sqrt{3/4}$	$acos\left(\sqrt{3/4}\right)$	A	$\sqrt{3}\,tan(z\sqrt{3}/2-\pi/6)$	c_n = $(-)^{n+1}$(2*A080635(n) $-\delta_{0,n}$)
-1, **-2**, 2, 2, -14, 10, 170, -670, -2270, 30490, -26950, -1435150, 8513650, 59564650, -1050090550, ...	$\sqrt{-5/4}$	$acos\left(\sqrt{5/4}\right)$	C	$-\sqrt{5}\,tanh\left(z\sqrt{5}/2+acosh(\sqrt{5}/2)\right)$	Not in OEIS
3, **3**, 9, 36, 189, 1242, 9801, 90234, 949401, 11237778, 147798189, 2138210946, ...	$\sqrt{-3/4}$	$acos\left(\sqrt{-3/6}\right)$	B	$-\sqrt{3}\,coth\left(z\sqrt{3}/2-acosh\left(\sqrt{3/2}\right)\right)$	Not in OEIS
3, **-3**, -9, -18, 27, 540, 2835, -1620, -183465, -1744740, -2205225, 176345100, 2703186675, ...	$-\sqrt{-15/4}$	$acos\left(\sqrt{15/6}\right)$	C	$-\sqrt{15}\,tanh\left(z\sqrt{15}/2-acosh\left(\sqrt{5/2}\right)\right)$	Not in OEIS
2, **3**, 6, 21, 96, 552, 3804, 30594, 281184, 2907408, 33402336, 422124816, 5819603136, ...	$\sqrt{2/4}$	$acos\left(\sqrt{2/6}\right)$	A	$\sqrt{2}\,tan\left(z/\sqrt{2}+acos(1/\sqrt{3})\right)$	Not in OEIS
2, **-3**, -6, -3, 48, 240, -60, -8670, -51360, 155760, 5037600, 29043600, -278116800, -6566138400, ...	$-\sqrt{-10/4}$	$acos\left(\sqrt{10/6}\right)$	C	$-\sqrt{10}\,tanh\left(z\sqrt{5/2}-acosh\left(\sqrt{5/3}\right)\right)$	Not in OEIS
1, **3**, 3, 12, 39, 210, 1155, 8130, 61995, 548490, 5296575, 56843850, 661572975, 8370001650, ...	$\sqrt{5/4}$	$acos\left(\sqrt{5/6}\right)$	A	$\sqrt{5}\,tan\left(z\sqrt{5}/2+acos\left(\sqrt{5/6}\right)\right)$	Not in OEIS
1, **-3**, -3, 6, 33, -12, -687, -1596, 20517, 150612, -627123, -13807164, -11222967, 1377176388,	$-\sqrt{-7/4}$	$acos\left(\sqrt{7/6}\right)$	C	$-\sqrt{7}\,tanh\left(z\sqrt{7}/2-acosh\left(\sqrt{7/6}\right)\right)$	Not in OEIS
0, **3**, 0, 9, 0, 108, 0, 2754, 0, 120528, 0, 8059824, 0, 764365248, 0, 97582435344, 0, 16135857600768, ...	$\sqrt{6/4}$	$acos\left(\sqrt{6/6}\right)$	A	$\sqrt{6}\cdot tan\left(z\sqrt{3/2}\right)$	Not in OEIS
0, **-3**, 0, 9, 0, -108, 0, 2754, 0, -120528, 0, 8059824, 0, -764365248, 0, 97582435344, 0, -16135857600768, ...	$-\sqrt{-6/4}$	$acos\left(\sqrt{6/6}\right)$	C	$-\sqrt{6}\,tanh\left(z\sqrt{3/2}\right)$	Not in OEIS
-1, **3**, -3, 12, -39, 210, -1155, 8130, -61995, 548490, -5296575, 56843850, -661572975, 8370001650, ...	$-\sqrt{5/4}$	$acos\left(\sqrt{5/6}\right)$	A	$\sqrt{5}\,tan\left(z\sqrt{5}/2-acos\left(\sqrt{5/6}\right)\right)$	Not in OEIS
-1, **-3**, 3, 6, -33, -12, 687, -1596, -20517, 150612, 627123, -13807164, 11222967, 1377176388, ...	$\sqrt{-7/4}$	$acos\left(\sqrt{7/6}\right)$	C	$-\sqrt{7}\,tanh\left(z\sqrt{7}/2+acosh\left(\sqrt{7/6}\right)\right)$	Not in OEIS

VII. Some sequences with rational starting coefficients

The extensions of the sequences $S\{c_0, c_1\}$ to rational pairs $\{c_0, c_1\}$, and their successive modification to make all their terms integer, are based on Lemma 7. To illustrate how it works, let us see a few very simple examples.

Example 7:

$$S\left\{\frac{1}{2},1\right\} = \{\frac{1}{2},\frac{1}{1},\frac{1}{2},\frac{5}{4},\frac{17}{8},\frac{109}{16},\frac{649}{32},\frac{5285}{64},\frac{44513}{128},\frac{448861}{256},\frac{4836601}{512},\frac{58743125}{1024},\dots\} \quad \text{(VII.7a)}$$

By Lemma 7, this sequence is related to $S\{1,4\}$. Specifically,

$$S_n\left\{\frac{1}{2},1\right\} = \frac{1}{2^{n+1}}S_n\{1,4\} \quad \text{(VII.7b)}$$

and, by equations (III.6a),

$$egf\left(z;S\left\{\frac{1}{2},1\right\}\right) = \frac{\sqrt{7}}{2}tan\left(\frac{\sqrt{7}}{4}z + acos\left(\sqrt{\frac{7}{8}}\right)\right). \quad \text{(VII.7c)}$$

Example 8:

$$S\left\{1,\frac{1}{2}\right\} = \left\{\frac{0!}{1},\frac{1!}{2},\frac{2!}{4},\frac{3!}{8},\frac{4!}{16},\frac{5!}{32},\dots\right\} \Longrightarrow S_n\left\{1,\frac{1}{2}\right\} = \frac{n!}{2^n} \quad \text{(VII.8a)}$$

Since in this case $c_1 = {c_0}^2/2$, this is one of the singular sequences and, by equation (III.8),

$$egf\left(z;S\left\{1,\frac{1}{2}\right\}\right) = 1/\left(1-\frac{z}{2}\right). \quad \text{(VII.8b)}$$

By Lemma 7, this sequence is related to the singular integer sequence $S\{2,2\}$ listed in Table I. Specifically,

$$S_n\left\{1,\frac{1}{2}\right\} = \frac{1}{2^{n+1}}S_n\{2,2\}. \quad \text{(VII.8c)}$$

Example 9:

$$S\left\{\frac{1}{2},\frac{1}{2}\right\} = \left\{\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{8},\frac{9}{16},\frac{39}{32},\frac{189}{64},\frac{1107}{128},\dots\right\} \quad \text{(VII.9a)}$$

By Lemma 7, this sequence is related to $S\{1,2\}$. Specifically,

$$S_n\left\{\frac{1}{2},\frac{1}{2}\right\} = \frac{1}{2^{n+1}}S_n\{1,2\} \quad \text{(VII.9b)}$$

$$egf\left(z; S\left\{\frac{1}{2}, \frac{1}{2}\right\}\right) = \frac{\sqrt{3}}{2} tan\left(z\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right). \quad \text{(VII.9c)}$$

VIII. Some related OEIS entries and their e.g.f.'s

It turns out that many of the integer sequences congruent with equations (I.1) or (I.3) are registered in OEIS, the Online Encyclopedia of Integer Sequences [7], often without mentioning any relationship between them. Realizing that they belong to the same family is a helpful classification feature (it also justifies registering a few more special cases and attempting a closer analysis of their shared properties).

The following is a short list of relevant OEIS entries. There must be many more; these are just those that were easy to spot without much effort.

A000111 [27]: In Example 1 we have already mentioned that this OEIS sequence matches exactly the sequence $S\{1,1\}$, i.e.,

$A000111(n) = S_n\{1,1\}$, and $egf(z; A000111) = egf(z; S\{1,1\}) = tan(z/2 + \pi/4)$.

The same e.g.f. formula appears in the OEIS entry in a note by Vaclav Kotesovec (Nov 8 2013). In another note in the same OEIS entry, Peter Bala (Sep 10 2015) remarks that the function satisfies the differential equation identical to our (I.3). Thus, in a sense, the OEIS entry $A000111$ anticipates this study, even if just for a single instance.

As discussed in Example 2, OEIS A000111 relates also to $S\{-1,1\}$ whose terms match those of $S\{1,1\}$, except for alternating signs:

$f(z) = egf(z; \{(-1)^n A000111(n)\}) = egf(z; S\{-1,1\}) = tan(z/2 - \pi/4)$, a function which also satisfies equation (I.3).

Not surprisingly, there are many more OEIS 'hits', even though not quite as close ones as OEIS A000111:

A000142 [28] is related to S{2,2}, in a rather trivial way, since

$A000142(n) = n! = S_n\{2,2\}/2 = 1/(1-z)$. An easy explicit evaluation shows that the function

$f(z) = 2 * A000142(n) = S_n\{2,2\} = 2/(1-z)$ indeed satisfies equation (I.3).

A000182 [29] is related to S{0,2} since, for $n > 0$, we have

$A000182(n) = S_{2n-1}\{0,2\}/2$. Considering that all even terms of S{0,2} are zero, it follows that

$\sum_{n=1}^{\infty} A000182(n)\frac{z^{2n-1}}{(2n-1)!} = egf(z; S\{0,2\})/2 = \tan(z)$, in agreement with the second e.g.f. formula in the OEIS A000182 entry.

A000629 [30] is related to S{3,4} by

$A000629(n) = \left(S_n\{3,4\} - \delta_{0,n}\right)/2$, where $\delta_{n,m}$ is the Kronecker delta. Hence

$egf(z;A000629) = (egf(z;S\{3,4\}) - 1)/2 = \left(coth\left(-z/2 + acosh\left(\sqrt{9/8}\right)\right) - 1\right)/2 = e^z/(2-e^z).$

The latter identity can be shown to hold through a somewhat tedious explicit manipulation or, more easily, using Wolfram Alpha. The final expression matches the e.g.f. reported in OEIS A000629.

The present insight indicates that $f(z) = 1 + 2 * egf(z;A000629) = (2+e^z)/(2-e^z)$ satisfies equation (I.3).

A002105 [31] is related to $S\{0,1\}$ since, for $n > 0$, we have

$A002105(n) = \{1,1,4,34,496,11056,349504,\dots\} = S_{2n-1}\{0,1\}$. Considering that all even terms of S{0,1} are zero, it follows that

$f(z) = \sum_{n=1}^{\infty} A002105(n)\frac{z^{2n-1}}{(2n-1)!} = egf(z;S\{0,1\}) = \sqrt{2}\,tan(z/\sqrt{2}),$

as noted by Michael Somos (Mar 05 2017) in the same OEIS entry. The function $f(z)$ satisfies equation (I.3).

A006673 [32] is related to $S\{2,-2\}$ through

$S_n\{2,-2\} = 2\delta_{n,0} - 2 * A006673(n)$. Consequently,

$egf(z;A006673) = 1 - egf(z;S\{2,-2\})/2 = 1 - \sqrt{2}\,tanh\left(-z\sqrt{2} + acosh(\sqrt{2})\right)$

This also proves that $f(z) = 2 - 2 * egf(z;A006673)$ satisfies equation (I.3).

A009764 [34]. As noted in the OEIS entry, this sequence is almost identical to OEIS A000182 which we have already discussed.

One can write $A009764(n) = A000182(n+1) - \delta_{0,n}$. Consequently,

$\sum_{n=0}^{\infty} A009764(n)\frac{z^{2n}}{(2n)!} = \frac{d}{dz}\left(\sum_{n=1}^{\infty} A000182(n)\frac{z^{2n-1}}{(2n-1)!}\right) - 1 = \frac{d}{dz}(\tan(z)) - 1,$

which indeed evaluates to $tan^2(z)$ as stated in the definition of A009764.

A080635 [35] is related to $S\{1,2\}$ and therefore (by equation VII.9b) to $S\{1/2,1/2\}$. In particular, we have

$S_n\{1,2\} = 2 * A080635(n) + \delta_{n,0}$, so that

$egf(z;A080635) = (egf(z;S\{1,2\}) - 1)/2 = (\sqrt{3}/2)\,tan(z\sqrt{3}/2 + \pi/6) - 1/2.$

For this sequence, the same result was obtained by slightly different means by Peter Bala (see his Sep 11 2015 comments in the OEIS entry), including the recurrence for the coefficients, and the differential equation for $egf(z; A080635)$ with a modified first term. We can rephrase his comment by saying that

$f(z) = 2 * egf(z; A080635) - 1$ satisfies equation (I.3).

A098558 [36] is another OEIS sequence closely related to $S\{2,2\}$, since

$A098558(n) = S_n\{2,2\} - \delta_{0,n}$, in agreement with

$egf(z; A098558) = egf(z; S\{2,2\}) - 1 = (1+z)/(1-z)$.

The function $f(z) = egf(z; A098558) + 1$ therefore satisfies equation (I.3).

A012509 [37] is related to S{0, 2} since, for $n > 0$, we have

$A012509(n) = S_{2n-1}\{0,2\}$. Considering that all even terms of S{0,2} are zero, it follows that

$\sum_{n=1}^{\infty} A012509(n) \frac{z^{2n-1}}{(2n-1)!} = egf(z; S\{0,2\}) = 2 * \tan(z)$.

A131178 [38] matches $S\{2,1\}$ except for the first element $S_0\{2,1\} = 2$ which is missing. Consequently,

$$egf(z; A131178) = egf(z; S\{2,1\}) - 2 = \sqrt{2} \cdot coth\left(-z\sqrt{2}/2 + acosh(\sqrt{2})\right) - 2 .$$

It proves that the function $f(z) = egf(z; A131178) + 2$ satisfies equation (I.3).

A208529 [39] matches the singular case $S\{2,2\}$ except for a shift in offsets: $S_n\{2,2\} = A208529(n+2)$.

Consequently, extending A208529 by setting A208529(0) = A208529(1) = 0, we have

$egf(z; A208529) = z^2 * egf(z; S\{2,2\}) = 2z^2/(1-z)$.

A230008 [40] is related to $S\{3,2\}$ through $S_n\{3,2\} = 2 * A230008(n) + \delta_{n,0}$. Hence

$$egf(z; A230008) = (egf(z; S\{3,2\}) - 1)/2 = \left(\sqrt{5}\, coth\left(-z\sqrt{5}/2 + acosh(3/2)\right) - 1\right)/2 ,$$

thus proving that the function $f(z) = 2 * egf(z; A230008) + 1$ satisfies equation (I.3).

A234797 [41] is related to $S\{1/2, 1\}$ and $S\{1,4\}$ since, for n>0,

$A234797(n) = 2^{n-1} S_n\{1/2, 1\} = S_n\{1,4\}/4$. Since the index offset of A234797 is 1, we can set A234797(0) = 0, so that

$$egf(z;\text{A234797}) = \frac{1}{4}(egf(z;S\{1,4\}) - 1) = \frac{\sqrt{7}}{4}\left(tan\left(\frac{\sqrt{7}}{2}z + acos\left(\sqrt{\frac{7}{8}}\right)\right) - 1\right)$$

and the function $f(z) = 4 * egf(z;\text{A234797}) + 1$ satisfies equation (I.3).

IX. Complex integer sequences satisfying (I.9) and their e.g.f.'s

When one or both of the starting terms c_0, c_1 is complex then, in general, the whole sequence $S\{c_0, c_1\}$ is complex. Another general rule is that when replacing $\{c_0, c_1\}$ by their complex conjugates, every element of the whole sequence gets replaced by its complex conjugates. It is also evident from the recurrence (I.9) that when the starting coefficients $\{c_0, c_1\}$ are complex integers, then so are all the elements of $S\{c_0, c_1\}$.

Some cases involving imaginary starting integers can be reduced to those with non-imaginary integers by applying Lemma 8, according to which

$$S_n\{jc_0, c_1\} = j^{n+1} S_n\{c_0, -c_1\} \quad \text{and} \quad egf(z;S\{jc_0, c_1\}) = j \cdot egf(jz;S\{c_0, -c_1\}). \qquad \text{(IX.1)}$$

These features limit considerably the number of qualitatively distinct[5] complex integer sequences, some typical examples of which are:

Example 10: For $S\{1, j\}$ we obtain the following sequence[6] of complex-valued integers:

$real(S\{1,j\}) = \{1, 0, 0, -1, -4, -11, -26, -23, 376, 4041, 28266, \ldots\}$

$imag(S\{1,j\}) = \{0, 1, 1, 1, 1, -3, -33, -179, -767, -2407, 863, \ldots\}$

Equations (III.6a) give

$\lambda_0 = +\sqrt{(2j-1)/4} = 0.393075\ldots + j\,0.636009\ldots$, $\lambda_1 = acos\left(\sqrt{1 + j/2}\right) = 0.452278\ldots - j\,0.530637\ldots$.

Consequently, the $egf(z;S\{1,j\})$ is neither purely trigonometric, nor purely hyperbolic. Rather, it is related to a 'cut' of the complex function $tan(z)$ along a generic straight line in the complex plane.

It is interesting to realize that there is an infinity of such generic linear 'cuts' whose generating functions have complex integer coefficients.

[5] We will say that two complex integer sequences are not qualitatively distinct if, for any n, their n-th coefficients differ only by one of the factors +1, -1 or j, –j.
[6] For clarity, let us list separately the real and imaginary parts of each of its terms.

Example 11: For $S\{0,j\}$ the resulting sequence is related to OEIS A002105 and A273352:

$$real(S\{0,j\}) = \{0,0,0,-1,0,0,0,\ 34,0,0,0,-11056,0,0,0,\ 14873104,\ 0,0,0,\ \ldots\}$$

$$imag(S\{0,j\}) = \{0,1,0,0,0,-4,0,0,0,\ 496,0,0,0,-349504,0,0,0,\ 819786496,\ldots\}$$

Equations (III.6a) give $\lambda_0 = \sqrt{j/2} = (1+j)/2, \lambda_1 = 0$, and therefore $egf(z; S\{0,j\}) = (1+j)\ tan(z\,(1+j)/2)$.
As expected, this is closely related to the sequence $S\{0,1\}$, the terms of which are "shuffled" between the real and imaginary parts.

Example 12: For $S\{0,2j\}$ the resulting sequence is:

$$real(S\{0,j\}) = \{1,0,0,-4,-16,-44,-104,\ 316,7456,67620,458280,1945036,\ \ldots\}$$

$$imag(S\{0,j\}) = \{0,2,2,2,2,-4,0,0,0,\ 496,0,0,0,-349504,0,0,0,\ 819786496,\ldots\}$$

Equations (III.6a) give in this case $\lambda_0 = \sqrt{j/2} = (1+j)/2, \lambda_1 = 0$, and therefore $egf(z; S\{0,j\}) = (1+j)\ tan(z\,(1+j)/2)$.

X. Conclusions and final remarks

The author's interest in the consequences of imposing relations (I.1) or (I.3) on a smooth real or complex functions started with the realization that many integer sequences conform to the resulting recursive definition (I.9) of their Taylor expansion coefficients. By itself, this is nothing new, of course; many people have studied the same thing without making much fuss about it. For me, I simply wanted to see the extent to which I could 'wrap up' the matter and as many of its consequences into a coherent piece of math.

It turned out to be quite simple in this case. The functions which satisfy

$$y''(z) = a.\,y'(z)y(z) \tag{X.1}$$

can be all expressed as

$$y(z) = (2\lambda_0/a)tan(\lambda_0 z + \lambda_1) \tag{X.2}$$

with some constant complex values of $\{\lambda_0, \lambda_1\}$ (integration constants), and their Taylor expansion can be written as

$$y(z) = (1/a)\sum_{n=0}^{\infty}(c_n/n!)z^n, \tag{X.3}$$

where the sequence of coefficients satisfies the recurrence relation (I.9) and is fully characterized by an arbitrary pair $\{c_0, c_1\}$ of starting complex values. However, there arise also special situations when $c_1 = {c_0}^2/2$, in which case the solution (X.2) degenerates into

$$y(z) = (c_0/a)/(1 - c_0 z/2), \tag{X.4}$$

an expression obtainable also by a suitable limit transition of the type $\lambda_0 \longrightarrow 0, \lambda_1 \longrightarrow \pi/2$.

Formulas were derived (III.6a and III.6b) which permit an easy conversion of the starting coefficients $\{c_0, c_1\}$ to/from the integration constants $\{\lambda_0, \lambda_1\}$.

From the recurrence relation it is evident that when $\{c_0, c_1\}$ are both real then the whole coefficients sequence is real-valued. Likewise, when the two starting coefficients are integers (or complex integers) then so are all of them. This means that any two integers define an integer sequence, and explains in part why there are so many entries in OEIS belonging to this family.

Appendix: PARI/GP scripts used in this study

The functions listed below are PARI/GP scripts used to prepare this article and numerically test most of the equations.
They make it possible to easily generate any of the sequences discussed here, cast in an OEIS-friendly format.

```
\\ Dependencies [42-44]: ------------------------------------
\\ The test functions such as IsEq(a,b) are available at
\\   https://oeis.org/wiki/File:EbUtils.txt
\\ The function IntSeq2OEIS(file,s,0)) is available at
\\   https://oeis.org/wiki/File:EbIo.txt

SequenceF2eqF1F(c0,c1,nmax=20,file="") =
/* ------------------------------------------------------------
Reference article: Stan Sykora, DOI 10.3247/SL6Math17.001.
Generates the sequence of coefficients defined by the
starting pair c0,c1 and the recurrence I.9 in the paper.
This is a general procedure; the coefficients may be integer,
rational, real, complex integer, or complex.
In any case, dumps also full info about the sequence and
its egf, as described in the article.
When file="" (default), the output is just displayed.
Otherwise, it is written into the specified plain text file.
When the starting coefficients are integer, then so are
all of them and the output file, after a brief header, has
```

```
a format compatible with an OEIS entry (including a b-file).
\\ *** ---------- Tests
The sections marked as Tests were used to make sure that all
steps were successfull (no problem was encountered). They
may be deleted.
---------------------------------------------------------- */
{
  my(C0=c0+0.0*I,C1=c1+0.0*I,cc0=C0,cc1=C1);
  my(sgn=1.0,l0,l1,r0,r1,rr0,rr1);

  print("TITLE: Sequence with c0=",c0,", c1=",c1,
      ", and e.g.f. f(z) satisfying f''=f'*f");
  if(file!="",
    write(file,"\nTITLE: Sequence with a(0)=",c0,", a(1)=",c1,
      ", and e.g.f. f(z) satisfying f''=f'*f"));

  if(IsEq(C1,0.0), error("Argument c1 may not be 0"));

  if(IsEq(C0,0.0),
    sgn=C1; sgn/=sqrt(sgn^2);
    l1=0;  \\ Special case c0=0
    l0=sgn*sqrt(C1/2),
    sgn=c0*c1; sgn/=sqrt(sgn^2);
    if(IsEq(2*C1,C0^2),
      l1=Pi/2;l0=0.0,  \\ Special case 2*c1=c0^2
        l1=acos(sqrt((2*C1-C0^2)/(2*C1)));  \\ Normal cases
        l0=sgn*sqrt((2*C1-C0^2)/4);
        cc0=2*l0*tan(l1);  \\ Backconversion
        cc1=2*(l0/cos(l1))^2)
  );

  \\ *** ---------- Tests
  if(!IsEq(C0,cc0),
    print(">>> Coefficient C0 back-conversion test failed: ",cc0);
```

```
  if(file!="",
     write(file,"!!! Coefficient C0 back-conversion test failed: ",cc0))
);
\\
if(!IsEq(C1,cc1),
  print(">>> Coefficient C1 back-conversion test failed: ",cc1);
  if(file!="",
     write(file,"!!! Coefficient C1 back-conversion test failed: ",cc1))
);
\\ *** ---------- End

if(IsEq(C0,0.0), r1=1; rr1=l1; r0=c1/2; rr0=l0,
  r1=(2*c1-c0^2)/(2*c1); rr1=acos(sqrt(r1));
    r0=(2*c1-c0^2)/4; rr0=sgn*sqrt(r0)
);

\\ ---------- Dumps
print("COMMENT: f(z) = 2*l0*tan(l0*z+l1), where:");
if(file!="",
  write(file,"\nCOMMENT: f(z) = 2*l0*tan(l0*z+l1), where:"));
\\
if(real(sgn)<0.0,
  print("l0 = -sqrt(",r0,") = ",l0);
  if(file!="",write(file,"l0 = -sqrt(",r0,")")),
  print("l0 = sqrt(",r0,") = ",l0);
  if(file!="",write(file,"l0 = sqrt(",r0,")")),
);
\\
print("l1 = acos(sqrt(",r1,")) = ",l1);
if(file!="",write(file,"l1 = acos(sqrt(",r1,"))","\n"));
\\ ----------

\\ *** ---------- Tests
if(!IsEq(l0,rr0),
```

```
    print(">>> Rational form of l0 failed: ",rr0);
    if(file!="",write(file,">>> Rational form of l0 failed: ",rr0)));
  \\
  if(!IsEq(l1,rr1),
    print(">>> Rational form of l1 failed: ",rr1);
    if(file!="",write(file,">>> Rational form of l1 failed: ",rr1)));
  \\ *** ---------- End

  my(s=vector(nmax+1));
  s[1]=c0;s[2]=c1;
  for(m=0,#s-3,s[m+3]=sum(k=0,m,binomial(m,k)*s[k+1]*s[m-k+2]));
  if(file!="",
    IntSeq2OEIS(file,s,0));
  return(s);
}
```

```
SequenceF2eqF1Fq(c0,c1,q=1,nmax=20,file="") =
/* ------------------------------------------------------------
Reference article: Stan Sykora, DOI 10.3247/SL6Math17.001
This version is similar to SequenceF2eqF1F, but includes the
power-multiplier q as used in Lemma 7, Equation II.4, i.e.,
generates the coefficients of egf(q*z;S{c0,c1})
------------------------------------------------------------ */
{
  my(C0=c0+0.0*I,C1=c1+0.0*I,cc0=C0,cc1=C1);
  my(sgn=1.0,l0,l1,r0,r1,rr0,rr1);

  if(IsEq(C1,0.0), error("Argument c1 may not be 0"));

  if(IsEq(C0,0.0),
    sgn=C1; sgn/=sqrt(sgn^2);
    l1=0;  \\ Special case c0=0
    l0=sgn*sqrt(C1/2),
    sgn=c0*c1; sgn/=sqrt(sgn^2);
```

```
   if(IsEq(2*C1,C0^2),
     l1=Pi/2;l0=0.0,  \\ Special case 2*c1=c0^2
       l1=acos(sqrt((2*C1-C0^2)/(2*C1)));  \\ Normal cases
       l0=sgn*sqrt((2*C1-C0^2)/4);
       cc0=2*l0*tan(l1);  \\ Backconversion
       cc1=2*(l0/cos(l1))^2)
 );

\\ *** ---------- Tests
if(!IsEq(C0,cc0),
   print(">>> Coefficient c0 back-conversion test failed: ",cc0);
   if(file!="",
      write(file,"!!! Coefficient c0 back-conversion test failed: ",cc0))
 );

 if(!IsEq(C1,cc1),
   print(">>> Coefficient c1 back-conversion test failed: ",cc1);
   if(file!="",
      write(file,"!!! Coefficient c1 back-conversion test failed: ",cc1))
 );
 \\ *** ---------- End

 if(!IsEq(q,1.0),
   print("q = ",q);
   if(file!="",write(file,"q = ",q));
 );

 if(IsEq(C0,0.0), r1=1; rr1=l1; r0=c1/2; rr0=l0,
   r1=(2*c1-c0^2)/(2*c1); rr1=acos(sqrt(r1));
     r0=(2*c1-c0^2)/4; rr0=sgn*sqrt(r0)
 );

 if(real(sgn)<0.0,
   print("l0 = -sqrt(",r0,") = ",l0);
```

```
    if(file!="",write(file,"l0 = -sqrt(",r0,") = ",l0)),
    print("l0 = sqrt(",r0,") = ",l0);
    if(file!="",write(file,"l0 = sqrt(",r0,") = ",l0)),
  );

  print("l1 = acos(sqrt(",r1,")) = ",l1);
  if(file!="",write(file,"l1 = acos(sqrt(",r1,")) = ",l1));

  \\ *** ---------- Tests
  if(!IsEq(l0,rr0),
    print(">>> Rational form of l0 failed: ",rr0);
    if(file!="",write(file,">>> Rational form of l0 failed: ",rr0)));

  if(!IsEq(l1,rr1),
    print(">>> Rational form of l1 failed: ",rr1);
    if(file!="",write(file,">>> Rational form of l1 failed: ",rr1)));
  \\ *** ---------- End

  my(s=vector(nmax+1));
  s[1]=c0;s[2]=c1;
  for(m=0,#s-3,s[m+3]=sum(k=0,m,binomial(m,k)*s[k+1]*s[m-k+2]));
  for(k=0,nmax,s[k+1]*=q^k);
  if(file!="",IntSeq2OEIS(file,s,0));
  return(s);
}
```

SequencesF2eqF1F(c1max=5,nmax=1000,path="c:") =

```
/* ----------------------------------------------------------
Reference article: Stan Sykora, DOI 10.3247/SL6Math17.001.
Generates the information comprized in Table I of the article.
Computes and dumps info about all integer sequences generated
by the recurrence I.9 for integer c0 and c1 values in the
following ranges: c1 = 1,..,c1max, c0 = -1,0,1,..,c1max.
For each sequence creates in the folder "path" a new file
```

```
named like "3,1.txt" or "3,-1.txt". The destination folder
must already exist.
---------------------------------------------------------- */
{
  my(file="");
  for(c1=1,c1max,
    for(c0=-1,c1max,
      file=Strprintf("%s%i,%i.txt",path,c0,c1);
      SequenceF2eqF1F(c0,c1,nmax,file);
      file=Strprintf("%s%i,%i.txt",path,c0,-c1);
      SequenceF2eqF1F(c0,-c1,nmax,file);
    )
  );
}
```

```
ConvertF2eqF1F_cl(c0,c1) =
/* ----------------------------------------------------------
Reference article: Stan Sykora, DOI 10.3247/SL6Math17.001
Uses the coefficients c0 and c1!=0 to compute and display
the lambda constants l0,l1 (global variables).
Embodies equations III.6a and III.6b of the article.
---------------------------------------------------------- */
{
  my(C0=c0+0.0*I,C1=c1+0.0*I,sgn=1.0);
  if(IsEq(C1,0.0), error("Argument c1 may not be 0"));
  if(IsEq(C0,0.0),
    sgn=C1; sgn/=sqrt(sgn^2); \\ Special case c0=0
    l1=0; l0=sgn*sqrt(C1/2),
    sgn=c0*c1; sgn/=sqrt(sgn^2);
    if(IsEq(2*C1,C0^2),
      l1=Pi/2;l0=0.0,  \\ Special case 2*c1=c0^2
      l1=acos(sqrt((2*C1-C0^2)/(2*C1)));  \\ Normal cases
      l0=sgn*sqrt((2*C1-C0^2)/4);
    )
```

```
  );
  print(l0," | ",l1);
}
```

```
ConvertF2eqF1F_lc(l0,l1) =
/* -----------------------------------------------------------
Reference article: Stan Sykora, DOI 10.3247/SL6Math17.001
Uses the lambda constants l0,l1 to compute and display
the coefficients c0,c1 (global variables).
Embodies equation III.5 of the article.
---------------------------------------------------------- */
{
  if(IsEq(l1,Pi/2), error("Argument l1 may not be Pi/2"));
  c0=2*l0*tan(l1);
  c1=2*(l0/cos(l1))^2;
}
```

```
TestF2eqF1F(C0,C1) =
/* -----------------------------------------------------------
Reference article: Stan Sykora, DOI 10.3247/SL6Math17.001
Uses the coefficients C0,C1 to compute the lambda constants,
then converts them back to the c-constants and compares them
back for an easy comparison. Used to test the conversion
formulas (full cycle).
---------------------------------------------------------- */
{
  c0=C0+0.0+0.0*I;
  c1=C1+0.0+0.0*I;
  print(c0," | ",c1);
  ConvertF2eqF1F_cl(c0,c1);
  ConvertF2eqF1F_lc(l0,l1);
  print(0.0+c0," | ",0.0+c1);
}
```

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History of this document

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