Fixed points of the mappings exp(z) and –exp(z) in *C*
With algorithms and 20-digit Tables

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This Note illustrates how even simple mappings often hide a non-intuitive inner richness. The functions exp(z) and its inverse log(z), for example, despite their apparent 'uneventful smoothness', define two denumerable sets of complex constants, namely the fixed points of the exponential mappings exp(z) and –exp(z). We analyze these two sets, including algorithms for their computation.

Keywords: mathematics, mapping, exponential function, logarithmic function, Lambert W function, complex numbers, fixed point, invariant, attractor

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Introduction

The exponential [1, 2] mapping[1] $E(a)$

$$E^+(a)\{z\} \equiv \exp(az), \quad (1)$$

with a being a constant, is certainly among the most ubiquitous ones in natural sciences as well as in engineering. One reason for this is the fact that its derivative is proportional to its value. It is therefore the simplest possible model for first-order evolution phenomena in which the rate of change of a time-dependent quantity is proportional to its current value. These include simple growths (expansions, explosions, ...), as well as decays (extinctions, relaxations, ...), all of utmost practical importance.

We do not dwell here on the various definitions of the exponential endomorphism in various domain sets such as real and complex numbers [3, 4], nor on their elementary properties which are amply treated in most calculus textbooks and other places. This Note addresses just the question of the existence and properties of fixed points of $E^+(1)$. Considering that fixed points [5] of any mapping[2] are among its most notable characteristics, and considering the importance of the exponential function, the interest in the set of its invariants appears more than justified.

In most practical applications, the constant a in equation (1) can be normalized to unity by a suitable change of scale (domain metric), which is the special case we are going to consider here:

$$E^+ : \quad E\{z\} = \exp(z). \quad (2a)$$

We will also consider a closely related endomorphism in $\boldsymbol{C}$, defined as

$$E^- : \quad E^-\{z\} = -\exp(z) \quad (2b)$$

or, more compactly,

$$E^\pm : \quad E^\pm\{z\} = \pm\exp(z) \quad (2c)$$

By definition, a fixed point[3] z_i of a function maps onto itself, which in our cases means:

$$\pm\exp(z_i) = z_i\,, \quad (3)$$

It is evident that E^+ has no solution in the domain of real numbers. In the domain $\boldsymbol{C}$ of complex numbers, however, such solutions exist (try, for example, z_1 =0.31813... + i*1.33723...) and we will see that they form a denumerable set.

Using $\bar{z}$ to denote the complex conjugate of z, and considering that, for any z in C,

$$\exp(\bar{z}) = \overline{\exp(z)}, \quad (4)$$

it is clear that when z_i satisfies either of the equations (3), so does $\overline{z_i}$. All fixed points of $E^\pm$ in $\boldsymbol{C}$ therefore come in conjugate pairs. We will see that the two conjugate members of each pair are distinct, with the exception of a single real-valued invariant of E^-.

From equations (3) it follows that, for any fixed point of $E^\pm$

$$z_i = Log(\pm z_i) = log(\pm z_i) + 2\pi K j, \quad (5)$$

where $Log(z)$ is the multivalued logarithmic function [6, 7, 8] in $\boldsymbol{C}$, $log(z)$ is its main branch, K is any integer (positive or negative), and j is the imaginary unit.

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[1] In general, $M(p_1, p_2, \dots)\{e\}$ denotes the result of the application of an endomorphism $M(p_1, p_2, \dots)$ in a set S to an element $e \in S$. The optional values $p_1, p_2, \dots$ are parameters specifying M in more detail.

[2] Also called invariant points (of a function), invariant elements (of a set), or just invariants (of a mapping).

[3] Throughout this Note, the subscript i stands for 'invariant' point/value. Other subscripts, such as k will be used for indexing purposes and assume integer values.

In fact, each of the mappings

$$L_K^{\pm} = log(\pm z) + 2\pi Kj \qquad (6)$$

is a right inverse of $E^{\pm}$. More specifically, we have

$$E^{\pm}\, L_K^{\pm}\,\{z\} = z \text{ for any integer } K\text{, while} \qquad (7a)$$

$$L_K^{\pm}\, E^{\pm}\,\{z\} = z + 2\pi Kj. \qquad (7b)$$

Let us now see the implications of equations (3). A complex number z can be written as

$$z = r.exp(i\varphi) = u + jv, \qquad (8)$$

where $\varphi = Arg(z) + 2\pi K$, with K being an integer denoting the complex plane *branch index*,

$$r = Abs(z) = \sqrt{u^2 + v^2},\;\; u = Real(z) = r.\cos(\varphi),\;\text{and}\;\; v = Imag(z) = r.\sin(\varphi)$$

are real-valued quantities (respectively azimuth, magnitude, real part and imaginary part).

For fixed points z_i of $E^{\pm}$, the defining equations (3) and (7) give

$$z_i \equiv u_i + jv_i = \pm\exp(u_i + jv_i) \equiv \pm\exp(u_i)\,(\cos(v_i) + j.sin(v_i)). \qquad (9)$$

This leads to the following constraints on the real-valued components u_i, v_i of z_i:

$$r_i^2 = u_i^2 + v_i^2 = \exp(2u_i), \qquad (10a)$$

$$v_i/u_i = sin(v_i)/cos(v_i) = tan(v_i), \qquad (10b)$$

$$sign(u_i) = \pm sign(cos(v_i)). \qquad (10c)$$

Equations (10a, 10b) allow us to formulate two functional constraints on u_i and v_i, each defining a curve[4] in the Cartesian complex plain (Figure 1):

$$v_i^2 = f(u_i),\;\text{with the real function } f(u) = \exp(2u) - u^2, \qquad (11a)$$

$$u_i = g(v_i),\;\text{with the real function } g(v) = v/tan(v). \qquad (11b)$$

Since v_i is real, the expression under the square root in equation (11a) must be non-negative, implying $u_i > -W(1) \cong -0.56714\ldots$ (OEIS [9] A030178 [10]), with W(x) denoting the Lambert W function [11, 12]. The fact that v_i appears in (11a) as a square underlines the already established fact that when $z_i$is an fixed point of either of the two mappings, so is also its complex conjugate. Hence, when $u_i > 0$, when taking the square root of $f(u_i)$ both signs are admissible.

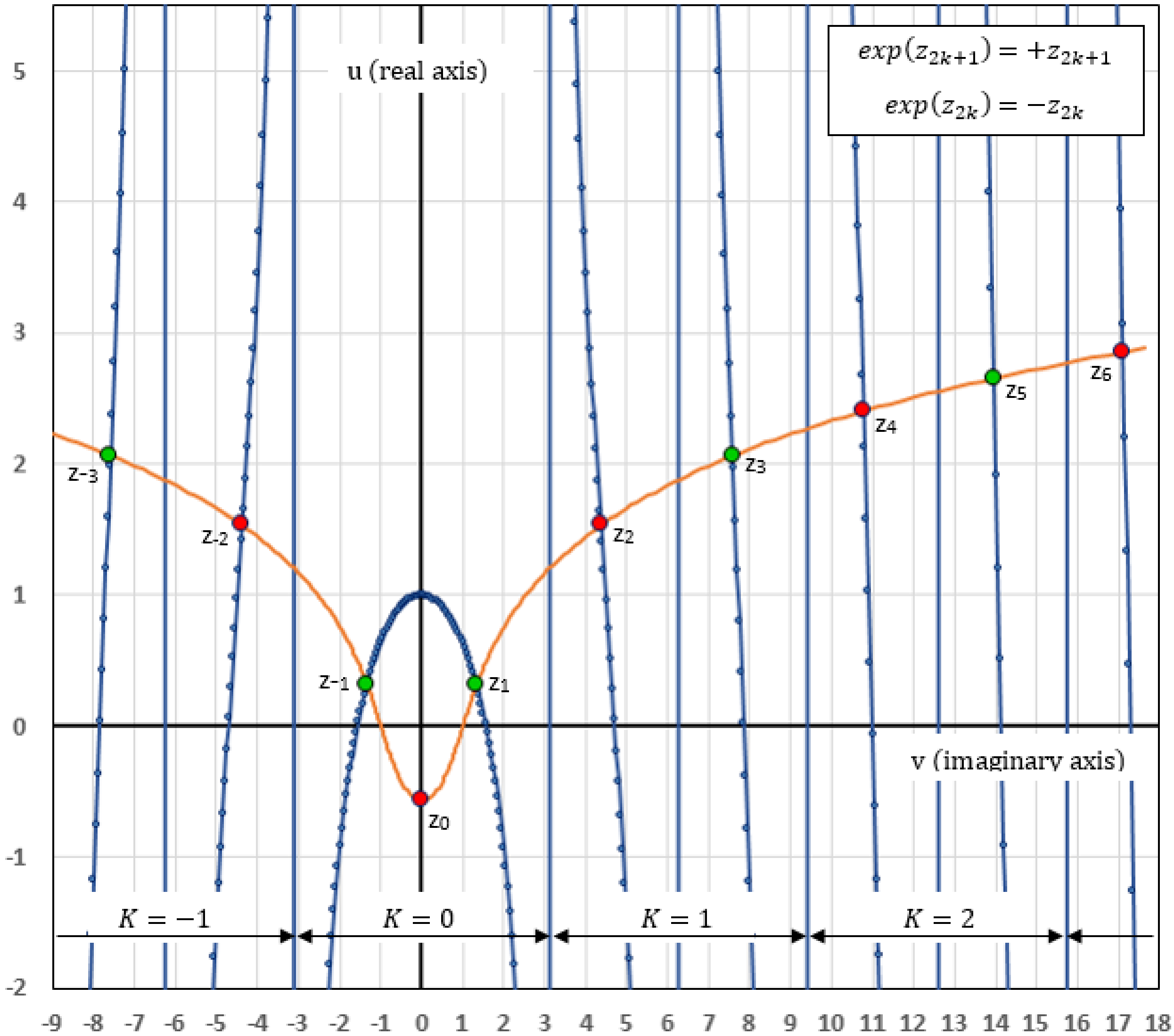
Any fixed point of $E^{\pm}$ with nonzero v_i must satisfy both (11a) and (11b). It is evident from Figure 1 that there indeed exists a denumerable set of pairs (u_i, v_i) which meet these conditions. For a full characterization, however, the constraint (10c) must be also taken into account. When $v_i$is non-zero then u_i must be positive, and (10c) implies that, for the fixed points of $E^{\pm}$, we require $\pm\cos(v_i) > 0$. Consequently, for E^+, v_i must lie in one of the intervals $(2\pi k - \pi/2\,, 2\pi k + \pi/2)$. As illustrated in Figure 1 (green dots), this eliminates half of the intersections between the functions $f(u)$ and $g(v)$. The remaining intersections (red dots) are those with v_i in an interval $(2\pi k + \pi/2\,, 2\pi k + 3\pi/2)$ and mark the fixed points of E^-.

There remains the special case when $v_i = 0$ in which (10b) is satisfied trivially and (11b) does not apply, (10a) implies a negative $u_i = -W(1)$, and (10c) indicates that it is an fixed point of E^- only (in Figure 1, the red dot marked as z_0).

[4] It may appear a bit unusual that for the first curve v is a function of u, while for the second one the two variables are interchanged. However, one can easily redefine the curves by means of a parameter t and the pair of equations [u(t)=t, v(t)=f(t)] for the first case, and [u(t)=t/tan(t), v(t)=t] for the second one.

Figure 1: Fixed points of E^+ (green dots) and E^- (red dots)

This graph displays the conventional Cartesian complex plane reflected about its diagonal, so that the real and imaginary axes are interchanged. The blue lines illustrate the curve $u = v/\tan(v)$ which has a singularity at every nonzero multiple of π, marked by a blue vertical line, and a zero at every half-integer multiple of π. The tiny blue dots are $g(v)$ points computed at regular intervals of v and displayed as a visual aide. The brown line illustrates the curve $v_i^2 = \exp(2u_i) - u_i^2$. Its value at $v = 0$ is $u = -W(1)$, corresponding to the only real-valued fixed point. All other fixed points of the two mappings correspond to the intersections between the two curves. There is an alternation between the fixed points of E^+(green dots, odd indices) and those of E^-(red dots, even indices). For more details, see the text.



We can now summarize the above analysis as follows:

The mappings $E^{\pm}$ have in $\boldsymbol{C}$ a denumerable set of fixed points denoted as $z_k, k = \cdots -2, -1, 0, 1, 2 \ldots$, with odd indexes marking the fixed points of E^+ and even indexes those of E^-. For any k, the values z_k, z_{-k} form a conjugate pair with distinct members, with the only exception of z_k which is real.

It is interesting to note how the branch index K kind of *'fades out of view'* once we get to the invariants of the mappings E^+ or E^-. The solutions of the equation $exp(z) = z$, for example, have all the same status because the function $exp(z)$ has no branches; only its right inverse $log(z)$ has them. The branch indices play a background role which, however, is important because it permits a neat organization and classification of all the solutions.

Relation to the Lambert function W

Rewriting equation (3) as $-(\pm 1) = -z_i \exp(-z_i)$, it is evident that the invariant elements are negated values of the multi-valued, complex Lambert $W(z)$ function [11] at $z = -1$ (for E^+) and $z = +1$ (for E^-).

By definition, $W(u)$ satisfies $W(u)exp\big(W(u)\big) = u$. Taking a logarithm and re-arranging a bit, we get $-W(u) = log(W(u)/u) + 2\pi lj$, with l being a generic integer yet to be defined[5]. Setting $u = -1$ and $W(-1) \equiv -z_k$, we recognize equation (5) for fixed points of E^+ (with odd index k). Similarly, setting $u = 1$ and $W(1) \equiv z_k$, we recognize equation (5) for fixed points of E^- (with even k). All these findings can be expressed in a compact way, valid for any index k, as

$$z_k = -W_L\big((-1)^k\big),\ \ L = -floor((k+1)/2), \tag{12}$$

where $W_L(z)$ denotes the L-th branch of $W(z)$. The chosen value of L is the one which guarantees a match between our fixed points numbering and the conventional indexing of the branches of Lambert W.

Asymptotic formulas

It is evident from Figure 1 that when $z_k = u_k + jv_k$ and $k > 0$ then, first of all,

$$(k-1)\pi \leq v_k \leq (k-1/2)\pi \tag{13}$$

More precisely, when $k \to \infty$, the value of v_k tends to the upper border of the above interval:

$$v_k \xrightarrow[k\to\infty]{} (k-1/2)\pi \tag{14}$$

Once the behavior of v_k is known, that of u_k can be deduced from equation (11a). Considering that u_k tends towards infinity, the term $\exp(2u)$ in the function $f(u)$ becomes soon dominant, so that

$$u_k \xrightarrow[k\to\infty]{} log(v_k) \xrightarrow[k\to\infty]{} log\big((k-1/2)\pi\big) \tag{15}$$

The convergence is very fast. For $k = 11$, for example, the ratio $log\big((k-1/2)\pi\big)/u_k$ is 0.99931..., in error by less than 0.0007.

Attractors of the mappings $L_K^{\pm}$ and the fixed points of $E^{\pm}$

A fixed point z_i of a mapping may (but need not) be its attractor [5], meaning that, starting from any point of an *attraction basin*, and applying the mapping iteratively, the consecutive images converge to z_i. However, this does not help us much in the case of the mapping E^+ because it does not have in $\boldsymbol{C}$ any

[5] Its presence, however, justifies to use of main-branch logarithm rather than the more generic multi-valued one.

attractor at all. One can see this empirically, trying to start at any randomly chosen point z in $\boldsymbol{C}$ and applying to it iteratively the function $exp(z)$. The result is that, after a few steps, the consecutive images start diverging, giving rise to an overflow. It is not difficult to prove that for E^+ this behavior is universal, but for the purposes of this Note, a simple statement of the empirically verifiable fact is sufficient.

The mapping E^- has in C one attractor which is the fixed point z_0 (the fixed points z_{2k} of E^- with nonzero k are not attractors of E^- as, again, one can easily ascertain empirically). The convergence of the sequence of consecutive images to z_0 upon iterated repetitions of E^- is not very fast. The distance to z_0 drops exponentially with a factor which has a limit of about 0.567... (curiously, it appears to converge to $|z_0|$). Depending on the choice of the initial point z_{ini}, the progression may start with a few apparently erratic looking steps before it settles into a smooth approach towards the attractor. For z_{ini} settings with large real values one may run into numerical overflow or underflow problems but those do not mean that the iterations would not eventually converge if sufficiently large precision were available.

Given any mapping M with a right inverse M_r^{-1}, it is evident from definitions that any invariant of the latter is also an invariant of the former. Moreover, should M_r^{-1} have an attractor, the attractor would be necessarily its fixed point, and therefore an invariant of M. In practice it often happens that, when M has an fixed point which is not an attractor, then the same value is an attractor of one of its right inverses.

Applied to our case, we see from equation (7a) that, for any K, $L_K^{\pm}$ is a right inverse of $E^{\pm}$. Therefore, should $L_K^{\pm}$have in $\boldsymbol{C}$ an attractor, the latter would be necessarily a fixed point of $E^{\pm}$. Extensive empirical tests[6] show that, in fact, the following is true:

For any non-zero integer K, each of the mappings L_K^+ and L_K^- has in $\boldsymbol{C}$ a unique attractor whose attraction region is the whole of $\boldsymbol{C}$. The attractors of L_K^- match our fixed points z_{2K}, those of L_K^+ match z_{2K+1} for positive K, and $z_{-(2K+1)}$ for negative K.

Moreover, the mapping L_0^+ has in $\boldsymbol{C}$ two mutually conjugate attractors matching our z_1 and z_{-1} whose attraction regions are, respectively, the upper and lower parts of the complex plane. Since $log(0)$ is undefined, however, one must avoid as starting points $z_{ini} = 0$ and all those real z_{ini} values which would end up at $z = 0$ upon a repetitive application of L_0^+, which are

0, 1, e, e^e, e^e^e, ... (16)

Finally, considering that, as discussed above, the mapping E^- has in $\boldsymbol{C}$ an attractor, it is not surprising that L_0^- does not have any.

Numeric evaluation

Armed with the knowledge of the attractors gained in the previous Section, it is now easy to code software functions for numeric evaluation of the fixed points. We do it using the free PARI/GP software [13] because it is simple, allows arbitrary precision, has intrinsically implemented unlimited-precision exponential and logarithmic functions, and is free. However, the code snippets listed below are properly commented to make clear the algorithm(s) which are anyway simple and easily portable to other programming languages.

The PARI functions listed in the box on the following page can be used to compute the value of any of the fixed points discussed above. Prior to calling these functions, one must set a desired default precision of the calculations and the global variable Eps_ used to interrupt the iterations. For example, to generate the

[6] A rigorous proof of the statement is relatively easy. In the context of this Note, however, I consider satisfactory extensive empirical tests which were carried out for various starting points over an a square region covering the interval of [-10,+10] on both real and imaginary axes, with a step of 0.1. The proof will be given for a more general situation, to be discussed in another document.

Tables in the next Section, we have configured the PARI/GP system to 20 digits precision by executing the following commands:

```
default(realprecision,20)
Eps_= 5*10.0^(-default(realprecision))
```

There are two functions, **ExpzEQz**(K) to compute an invariant of E^+ (namely, the solution of the equation $exp(z) = z$ corresponding to the K-th branch of $log(z)$), and **ExpzEQmz**(K) to compute an invariant of E^-(namely, the solution of $exp(z) = -z$ for the same branch of $log(z)$).

```
ExpzEQz(K,sgn=1) = {
/* ----------------------------------------------------------
Solves for exp(z)= z in the K-th branch of log(z).
Set the optional second argument to -1 to select the
solution with negative imaginary part in case of K=0
(it has no effect when K is nonzero).
Prior to calling this function, make sure that the
Global variable Eps_ is set to the desired precision,
compatibly with default(realprecision).
------------------------------------------------------- */
  my(z=1+sgn*I,zlast=z,ncyc=1); \\ z_ini is set to 1+I
  while(ncyc, \\ The cycle will be terminated by a 'break'
    z=log(zlast)+2*Pi*K*I; \\ Apply the mapping (L+)_K
    if(abs(z-zlast)<Eps_,break); \\ Test for termination
    zlast=z;ncyc++); \\ Proceed to next iteration
  \\ Uncomment next two lines to play with convergence rate
  \\ print("Cycles: ",ncyc);
  \\ print("Convergence factor per cycle: ",Eps_^(1.0/ncyc));
  return(z);
}

ExpzEQmz(K) = {
/* ----------------------------------------------------------
Solves for exp(z)= -z in the K-th branch of log(z).
Prior to calling this function, make sure that the
Global variable Eps_ is set to the desired precision,
compatibly with default(realprecision).
------------------------------------------------------- */
  my(z=-1.0,zlast=z,ncyc=1); \\ z_ini is set to -1
  while(ncyc, \\ The cycle will be terminated by a 'break'
    if(K,z=log(-z)+2*Pi*K*I, \\ K!=0: apply mapping (L-)_K
       z=-exp(z)); \\ K==0: apply -exp(z)
    if(abs(z-zlast)<Eps_,break); \\ Test for termination
    zlast=z;ncyc++); \\ Proceed to next iteration
  \\ Uncomment next two lines to play with convergence rate
  \\ print("Cycles: ",ncyc);
  \\ print("Convergence factor per cycle: ",Eps_^(1.0/ncyc));
  return(z);
}
```

Computational performance:

Both functions use a simple iterative procedure to approach the desired attractor – and fixed point - with the specified precision. As is usual in algorithms of this type, the convergence, after a few atypical points, is exponential in the sense that the distance of the current value from the attractor drops by an approximately constant factor $c < 1$ in every iteration. Algorithms of this type are usually classified as 'efficient', even though it is the slowest convergence type to which the label can be applied.

One can easily experiment with the above code and, in particular, find out the number of steps required to reach a given precision and/or the value of the convergence factor c. It turns out that these features depend relatively little on the choice of the initial value z_{ini} and they differ somewhat (not too much) between the two mappings. They are strongly dependent, however, on the branch index K, with the efficiency increasing sharply with increasing absolute value of the branch index (see Table I).

Table I: Selected convergence data obtained with 100 digits precision
The rounded c values are averages over the whole runs.

K	Default z_{ini}				$z_{ini} = 1234 + 4321j$	
	ExpzEQz		ExpzEQmz		ExpzEQz	ExpzEQmz
	Cycles	c factor	Cycles	c factor	Cycles	Cycles
0	832	0.72802	466	0.56737	838	overflow
1	131	0.13318	176	0.22300	132	176
2	102	0.07508	113	0.09660	102	113
4	83	0.04150	86	0.04638	83	87
8	69	0.02176	70	0.02298	69	71
16	59	0.01137	60	0.01226	59	60
32	52	0.00622	52	0.00622	52	52
64	46	0.00321	46	0.00321	46	46

One might think that a more realistic initial estimate for z_{ini} would reduce the number of required cycles. It does, but not much, the performance boost is not significant. Typically, one can save four cycles for very low K and not more than one, or even none, for K values exceeding 3 or so. For the present purposes, therefore, a better z_{ini} estimate is irrelevant.

The convergence can be improved using various 'tricks'. For example, replacing the mapping $L_0^+ = log(z)$ with $(z + 3 * log(z))/4$ drops the c factor for ExpzEQz(0) from 0.728 to 0.651, and the number of cycles from 832 to 616 (for the 100-digits precision). However, the same modification is counterproductive for $|K| = 1$ and, increasingly so, for all higher values of K.

More powerful attractor convergence-acceleration methods exist but they are beyond the scope and focus of this investigation. Knowing that we can compute any of the fixed points to any desired precision within a few seconds[7] is more than sufficient in the present context.

[7] On my (slow) PC, evaluating z_1 to 1000 digits (using the unmodified algorithm) takes about 7 seconds, and substantially less for the higher invariants. Any evaluation with 20-digits precision takes at most a fraction of a millisecond.

Table II: First 100 fixed points of the mapping exp(z)

All these $z_k$values with odd k are solutions of the equation $exp(z) = z$, and values $W_K(-1)$ of the Lambert W function in its K-th branch. These properties are shared also by their complex conjugates corresponding to negated values of k and K (hence, the Table covers 200 fixed points). The values are *rounded* to 20 significant digits.

K	k	Real part of z_k	Imaginary part of z_k	Modulus (abs. value)
0	1	0.31813150520476413 53	1.3372357014306894089	1.3745570107437074865
1	3	2.0622777295982838850	7.5886311784725126226	7.8638611760942326688
2	5	2.6531919740386972866	13.949208334533214455	14.199290151670056423
3	7	3.0202397081645011514	20.272457641615221810	20.496204202872711234
4	9	3.2877686115440937515	26.580471499359145698	26.783033576712299097
5	11	3.4985152121541032651	32.880721480068912759	33.066319023131964923
6	13	3.6724500687098179322	39.176440021735248576	39.348193634318031682
7	15	3.8205543078136768665	45.469265403710858577	45.629494097261815144
8	17	3.9495227424225290276	51.760122004020700577	51.910586201313703517
9	19	4.0637417027918296891	58.049573434477498900	58.191640057243662522
10	21	4.1662424475284168612	64.337984120359044986	64.472736693917066956
11	23	4.2592078559390358387	70.625600802137234815	70.753913956917891655
12	25	4.3442623028349102442	76.912596859781745855	77.035187873277373807
13	27	4.4226473672790146136	83.199097908843236152	83.316563194724870684
14	29	4.4953334317186493053	89.485197323844476396	89.598038833167046103
15	31	4.5630933498256177846	95.770966045047388159	95.879610752864670458
16	33	4.6265526777907663625	102.05645899156936319	102.16127354128986205
17	35	4.6862248854570079366	108.34171938138198275	108.44302127011768619
18	37	4.7425366350665273407	114.62678171714661533	114.72484796488215741
19	39	4.7958463114371479629	120.91167389694539266	121.00674785483738111
20	41	4.8464578566848846867	127.19641873639019421	127.28871549795660145
21	43	4.8946312688228741701	133.48103508588498562	133.57074583477101020
22	45	4.9405906854558158013	139.76553866384870535	139.85283420194601160
23	47	4.9845306899927294777	146.04994268706757471	146.13497632358655411
24	49	5.0266212897345808108	152.33425835379593357	152.41716829081774774
25	51	5.0670118879905553236	158.61849521840106895	158.69940653582470276
26	53	5.1058344847351415135	164.90266148505363712	164.98168780394902598
27	55	5.1432062789111839778	171.18676424024908290	171.26400912589327096
28	57	5.1792318017940729570	177.47080963858814112	177.54636779115675430
29	59	5.2140046793078506666	183.75480305246963217	183.82876132326994226
30	61	5.2476090981418515602	190.03874919365249244	190.11118745706381227
31	63	5.2801210334721515443	196.32265221269375744	196.39364411801728192
32	65	5.3116092833443259707	202.60651578084207299	202.67612940361654139
33	67	5.3421363451414094768	208.89034315791056874	208.95864156660215897
34	69	5.3717591622123910223	215.17413724886400359	215.24117899995292545
35	71	5.4005297630802567021	221.45790065125995818	221.52374022344702525
36	73	5.4284958112583310149	227.74163569523091469	227.80632387164348681
37	75	5.4557010802693059180	234.02534447734647888	234.08892868313515300
38	77	5.4821858657549232642	240.30902888942616813	240.37155349093566094
39	79	5.5079873444166780900	246.59269064316373370	246.65419721387533102
40	81	5.5331398878125835450	252.87633129125965415	252.93685884889337171
41	83	5.5576753376564941299	259.15995224562865855	259.21953746412580544
42	85	5.5816232481521108739	265.44355479314600886	265.50223219269968996
43	87	5.6050110999879676582	271.72714010931383374	271.78494222715440780
44	89	5.6278644898795157909	278.01070927016253876	278.06766681441998568

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45	91	5.6502072989365675449	284.29426326264876516	284.35040525129061150
46	93	5.6720618426327954479	290.57780299376787101	290.63315688033879808
47	95	5.6934490047382487703	296.86132929856340620	296.91592108622207589
48	97	5.7143883572297640684	303.14484294718694227	303.19869729233976360
49	99	5.7348982679048538539	309.42834465113764351	309.48148495780234641
50	101	5.7549959971818823835	315.71183506879113748	315.76428357468036632
51	103	5.7746977853648307282	321.99531481031077776	322.04709266550356607
52	105	5.7940189314780593586	328.27878444202066622	328.32991178098439584
53	107	5.8129738646298018831	334.56224449030831749	334.61274049794294597
54	109	5.8315762087382781832	340.84569544511520531	340.89557841741296268
55	111	5.8498388413477107527	347.12913776306530511	347.17842516291088157
56	113	5.8677739471702199474	353.41257187027487876	353.46128037885181647
57	115	5.8853930669111363430	359.69599816488092296	359.74414372909820305
58	117	5.9027071418676957738	365.97941701932074828	366.02701489562834901
59	119	5.9197265547327105752	372.26282878239092930	372.30989357731351068
60	121	5.9364611669842532857	378.54623378111025031	378.59277948879332434
61	123	5.9529203531984889287	384.82963232240816926	384.87567235944048843
62	125	5.9691130325845787667	391.11302469465765445	391.15857193240653729
63	127	5.9850476980072365210	397.39641116906894784	397.44147796374138499
64	129	6.0007324427333598115	403.67979200095882009	403.72439022158006011
65	131	6.0161749851136035185	409.96316743090815901	410.00730848539071251
66	133	6.0313826913873148771	416.24653768581923681	416.29023254527855980
67	135	6.0463625967794932027	422.52990297988269820	422.57316220134096432
68	137	6.0611214250410129013	428.81326351546317559	428.85609726306929706
69	139	6.0756656065679515036	435.09661948391144383	435.13903754879366162
70	141	6.0900012952222333118	441.37997106631015708	441.42198288516692302
71	143	6.1041343839637092534	447.66331843415944641	447.70493310668481938
72	145	6.1180705193930514150	453.94666175000798563	453.98788805523923308
73	147	6.1318151152952797313	460.23000116803454094	460.27084757970196540
74	149	6.1453733652652143244	466.51333683458449792	466.55381153553659977
75	151	6.1587502544885351661	472.79666888866539806	472.83677978443625481
76	153	6.1719505707453227152	479.07999746240510787	479.11975219398522321
77	155	6.1849789146968547119	485.36332268147588124	485.40272863734266848
78	157	6.1978397095109632647	491.64664466548725340	491.68570899294670969
79	159	6.2105372098763410461	497.92996352835041822	497.96869314423736800
80	161	6.2230755104517631140	504.21327937861648478	504.25168097939697812
81	163	6.2354585537922066422	510.49659231979078146	510.53467239110678513
82	165	6.2476901377902564721	516.77990245062517144	516.81766727631855378
83	167	6.2597739226679374667	523.06320986539016125	523.10066553604011338
84	169	6.2717134375511778494	529.34651465412842044	529.38366707513384954
85	171	6.2835120866564480887	535.62981690289118334	535.66667180212723338
86	173	6.2951731551167083801	541.91311669395887211	541.94967962903455162
87	175	6.3066998144716086202	548.19641410604716135	548.23269047118906677
88	177	6.3180951278448951759	554.47970921449959770	554.51570424708489718
89	179	6.3293620548301684816	560.76300209146779100	560.79872087822796150
90	181	6.3405034561044865714	567.04629280608010672	567.08174028899538243
91	183	6.3515220977878060704	573.32958142459971014	573.36476240650279067
92	185	6.3624206555648796759	579.61286801057274181	579.64778716047901173
93	187	6.3732017185849750345	585.89615262496733851	585.93081448314765722
94	189	6.3838677931536328039	592.17943532630415590	592.21384430911517715
95	191	6.3944213062296314142	598.46271617077899508	598.49687657526496230
96	193	6.4048646087393635138	604.74599521237808738	604.77991122065711569
97	195	6.4151999787199461308	611.02927250298654704	611.06294818643353904
98	197	6.4254296243015758888	617.31254809249046163	617.34598741572800585
99	199	6.4355556865388955847	623.59582202887305318	623.62902885358091530

Table III: First 100 fixed points of the mapping -exp(z)

All these $z_k$values with even k are solutions of the equation $exp(z) = -z$, and values $W_K(1)$ of the Lambert W function in its K-th branch. These properties are shared also by their complex conjugates corresponding to negated values of k and K. The values are *rounded* to 20 significant digits.

K	k	Real part of z_k	Imaginary part of z_k	Modulus (abs. value)
0	0	-.56714329040978387300	0.0	0.56714329040978387300
1	2	1.5339133197935745079	4.3751851530618983855	4.6362846327866251895
2	4	2.4015851048680028842	10.776299516115070898	11.040663126685179665
3	6	2.8535817554090378072	17.113535539412145913	17.349813471433227303
4	8	3.1629527388040840093	23.427747503755212819	23.640296595593230026
5	10	3.3986921967647194819	29.731310707828526210	29.924938513784595158
6	12	3.5892625245295749005	36.029021703427674892	36.207364035180120433
7	14	3.7492425412169807420	42.323145361236994865	42.488886228062835869
8	16	3.8871164495491617985	48.614898564936282096	48.770052663201005218
9	18	4.0082620531092576890	54.904997123349749065	55.051114674485011113
10	20	4.1163046640017699831	61.193891331956510477	61.332179974579289957
11	22	4.2138049147167743704	67.481879520015322941	67.613313891667480627
12	24	4.3026389193033564509	73.769167656040994602	73.894538352539890299
13	26	4.3842225073788582586	80.055902804540732194	80.175862831928912352
14	28	4.4596505195112867401	86.342192948825070346	86.457288680413260959
15	30	4.5297870804820397976	92.628119271810462660	92.738813076441984604
16	32	4.5953262041331040497	98.913744054924693970	99.020431153805539279
17	34	4.6568337148901730335	105.19911592355408442	105.30213716418821256
18	36	4.7147769993763328246	111.48427342777463854	111.58392511411608544
19	38	4.7695465967269711403	117.76924754616840906	117.86578910917723828
20	40	4.8214721753602549192	124.05406347404712454	124.14772353273613724
21	42	4.8708345616349407896	130.33874192479432236	130.42972313036992191
22	44	4.9178749363625923611	136.62330009289826365	136.71178304068686994
23	46	4.9628019635879969342	142.90775237744401410	142.99389879607677913
24	48	5.0057973856476142074	149.19211093309842126	149.27606630716158435
25	50	5.0470204642215697094	155.47638609494158672	155.55828183902526276
26	52	5.0866115417413472166	161.76058670974591975	161.84054198394728735
27	54	5.1246949243013037730	168.04472039698825060	168.12284363336565493
28	56	5.1613812355204892332	174.32879375646336395	174.40518395059799069
29	58	5.1967693537516344162	180.61281253487824132	180.68756034513046397
30	60	5.2309480181242990694	186.89678176062119523	186.96997044885466315
31	62	5.2639971691199272642	193.18070585361043732	193.25241209437894229
32	64	5.2959890746558078217	199.46458871546065663	199.53488329539461357
33	66	5.3269892815873713487	205.74843380398031330	205.81738222899691626
34	68	5.3570574241345213790	212.03224419510083221	212.09990721982021968
35	70	5.3862479142974862234	218.31602263465445024	218.38245672583049377
36	72	5.4146105343494619033	224.59977158189883753	224.66502932561582984
37	74	5.4421909476139879136	230.88349324629023289	230.94762370702158671
38	76	5.4690311406889710131	237.16718961870144044	237.23023865698679099
39	78	5.4951698078704236249	243.45086249804395310	243.51287305245042078
40	80	5.5206426866111739794	249.73451351406809815	249.79552585220876007
41	82	5.5454828513132150239	256.01814414696918045	256.07819608961731863
42	84	5.5697209715137988182	262.30175574431199743	262.36088286604242282
43	86	5.5933855395213576190	268.58534953569396135	268.64358534497827657
44	88	5.6165030717390311774	274.86892664549320378	274.92630274675498519
45	90	5.6390982872432236983	281.15248810398850364	281.20903434377172676
46	92	5.6611942666327753584	287.43603485708964603	287.49177945619699026

47	94	5.6828125937079478494	293.71956777487754442	293.77453744808464749
48	96	5.7039734821593086487	300.00308765912132896	300.05730772386066454
49	98	5.7246958891303409298	306.28659524991320002	306.34008972514057330
50	100	5.7449976172527183817	312.57009123154005419	312.62288292784249110
51	102	5.7648954065304659987	318.85357623769282991	318.90568683956457433
52	104	5.7844050172612737544	325.13705085609949437	325.18850099719938643
53	106	5.8035413050240519794	331.42051563265504320	331.47132496476081475
54	108	5.8223182886265517530	337.70397107511136602	337.75415833140193830
55	110	5.8407492117915537339	343.98741765638098264	344.03700070960467938
56	112	5.8588465992615056463	350.27085581750118728	350.31985173352420714
57	114	5.8766223079168907406	356.55428597029881566	356.60271105747293981
58	116	5.8940875734308322227	362.83770849979048016	362.88557835453064558
59	118	5.9112530529196556200	369.12112376634854333	369.16845331526859946
60	120	5.9281288639948243096	375.40453210765919274	375.45133564657703892
61	122	5.9447246205745668674	381.68793384049563150	381.73422507058629691
62	124	5.9610494657725691389	387.97132926232652352	388.01712132367299499
63	126	5.9771121021454197615	394.25471865277735527	394.30002415554356887
64	128	5.9929208195493266585	400.53810227496023775	400.58293332838818703
65	130	6.0084835208293316624	406.82148037668582148	406.86584861609882305
66	132	6.0238077455403056572	413.10485319156939181	413.14876980354586468
67	134	6.0389006918779530029	419.38822094004181594	419.43169668590819643
68	136	6.0537692369795046333	425.67158383027479650	425.71462906805218573
69	138	6.0684199557374038951	431.95494205902882452	431.99756676395544359
70	140	6.0828591382548054051	438.23829581243129499	438.28050959617162366
71	142	6.0970928060588726256	444.52164526669143316	444.56345739533287587
72	144	6.1111267271764662043	450.80499058875796414	450.84640999968688579
73	146	6.1249664301666829046	457.08833193692482778	457.12936725466571394
74	148	6.1386172171956792069	463.37166946138968546	463.41232901248390249
75	150	6.1520841762311602720	469.65500330476947457	469.69529513176354566
76	152	6.1653721924267176519	475.93833360257683212	475.97826547718422472
77	154	6.1784859587597568663	482.22166048366082387	482.26123991915589404
78	156	6.1914299859809807831	488.50498407061507394	488.54421833351297126
79	158	6.2042086119282099270	494.78830448015608563	494.82720060122803532
80	160	6.2168260102526593603	501.07162182347427368	501.11018660814367245
81	162	6.2292861986015949348	507.35493620655998689	507.39317624472113333
82	164	6.2415930462975080609	513.63824773050658429	513.67616940580457659
83	166	6.2537502815505324494	519.92155649179243510	519.95916599039977476
84	168	6.2657614982377388000	526.20486258254354013	526.24216590146625104
85	170	6.2776301622801490603	532.48816609077831721	532.52516904572189881
86	172	6.2893596176457797309	538.77146710063595389	538.80817533345921151
87	174	6.3009530920047263843	545.05476569258960570	545.09118467837232026
88	176	6.3124137020602148725	551.33806194364560536	551.37419699739409922
89	178	6.3237444585776471489	557.62135592752974670	557.65721221054265647
90	180	6.3349482711319421499	563.90464771486161541	563.94023024077658089
91	182	6.3460279525918978136	570.18793737331785561	570.22325101385836310
92	184	6.3569862233588639544	576.47122496778518647	576.50627445822545302
93	186	6.3678257153757039076	582.75451056050391490	582.78930050486845650
94	188	6.3785489759208235752	589.03779421120262882	589.07232908721601042
95	190	6.3891584712009490132	595.32107597722469961	595.35536014102590944
96	192	6.3996565897553283716	601.60435591364717136	601.63839360428208886
97	194	6.4100456456831122258	607.88763407339256846	607.92142941709709615
98	196	6.4203278817048203786	614.17091050733411083	614.20446752161971035
99	198	6.4305054720680261337	620.45418526439478775	620.48750786194739232

Final remarks

We have analyzed the structure and some of the properties of the denumerable set of points in $\boldsymbol{C}$ which are mapped onto themselves under the exponential mappings $\pm exp(z)$. It turns out that the separate denumerable sets of fixed points of the two mappings are closely related and, to classify them and get a clear view of their relationships, it is best to study them together as a single construct.

By-products of the analysis:

- We will explored in more detail elsewhere the numerical evaluation of the Lambert W_K function in all its branches (indexed by the integer K). The fixed points which were discussed here regard only the values $W_K(\pm 1)$, but a generalization to any other value is easy to envisage.

- The fixed points z_k are simple poles of the following functions

For odd k, $\quad s(z) = 1/(exp(z) - z)$ (17a)

For even k $\quad t(z) = 1/(exp(z) + z)$. (17b)

The first of these is particularly interesting since it has no singularity on the real axis, nor on the imaginary axis. When considered as a real function of real variable, it looks as a peak with a maximum at the origin, an asymptotically exponential decay for positive arguments, and an asymptotic decay of the $1/x$ type for negative arguments. The convergence radius of its Taylor expansion is $|z_1|$. A spectroscopist, for example, might interpret it as an asymmetric spectral peak and compare it with the symmetric Lorentzian peak shape which is ubiquitous in spectroscopy. What might surprise him is the fact that all its complex poles (the z_k with odd k) have their real parts positive and are therefore shifted away to one side of the location of its maximum. The second function has for real arguments a singularity at z_0 (a negative value) but no singularity along the imaginary axis. Both functions, shown in Figure 2, look like they might merit further investigation.

- The fixed points are also complex solutions of the following equations:

For odd k: $\quad z^2 - 2z * cosh(z) + 1 = 0, \;\; z^2 - 2z * sinh(z) - 1 = 0.$ (18a)

For even k: $\quad z^2 + 2z * cosh(z) + 1 = 0, \;\; z^2 + 2z * sinh(z) - 1 = 0.$ (18b)

S. Sykora, *Fixed points of exp(z) and –exp(z) in C*, DOI: 10.3247/SL6Math16.002

Appendix

PARI/GP programs were written to automate the generation of the Tables II and III. These are their listings:

```
Tab_ExpzFixed(Kmax,file) = {
/* ----------------------------------------------------------
Lists fixed points of the mapping exp(z) with positive
imaginary parts, one corresponding to each K, the branch
index of log(z). Generates a 5-column plain text file with
one line for every K, ranging from 0 to Kmax.
The columns are: K, k=2*K+1, followed by the z_k values,
namely its real part, imaginary part, and absolute value.
---------------------------------------------------------- */
SetEbDefaults(20);
for(K=0,Kmax,z=ExpzEQz(K);
    write(file,K,"\t",2*K+1,"\t",
    real(z),"\t",imag(z),"\t",abs(z)));
}

Tab_ExpmzFixed(Kmax,file) = {
/* ----------------------------------------------------------
Lists fixed points of the mapping -exp(z) with positive
imaginary parts, one corresponding to each K, the branch
index of log(z). Generates a 5-column plain text file with
one line for every K, ranging from 0 to Kmax.
The columns are: K, k=2*K, followed by the z_k values,
namely its real part, imaginary part, and absolute value.
---------------------------------------------------------- */
SetEbDefaults(20);
for(K=0,Kmax,z=ExpzEQmz(K);
    write(file,K,"\t",2*K,"\t",
             real(z),"\t",imag(z),"\t",abs(z)));
}
```

OEIS registrations

Several of the z_k values were present in OEIS [9] before this Note was written, registered in contexts related to the present topic but stopping at the very first steps, congruent with just the main branch of the logarithmic function ($K = 0, |k| \leq 1$). These are:

z_0 A030178 (negated)
z_1 A059526 (real part), A059527 (imaginary part) , A238274 (modulus)

Submitted, but pending:

A few more registrations will be made[8] by the Author:
z_2 A276759 (real part), A276760 (imaginary part) , A276761 (modulus)
z_3 A277681 (real part), A277682 (imaginary part) , A277683 (modulus)

[8] 23 Nov 2016: now approved

References and Links

[1] Wikipedia: Exponential function.
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[9] N. J. A. Sloane, editor, OEIS, The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org.
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