Binary Iterated Powers

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Abstract

Binary iterated powers (bips) are defined as functions of the type

$f(x) = \beta\text{^}(e_0\beta\text{^}(e_1\beta\text{^}(e_2\beta\text{^}(...\beta\text{^}(e_n x)...)))) \equiv \{b_0b_1b_2...b_n|x\}$, where β is a non-negative *base*,

$e_k = 2b_k-1$, and $b = \{ b_0,b_1,b_2,... ,b_n\}$ is a binary sequence of 0's and 1's.

This paper explores the properties of both finite and infinite binary iterated powers in the real and complex domains. In particular, it analyses the convergence behavior of infinite bips and shows that, for a range of bases and *any* infinite binary sequence b, they converge to a real value which depends upon b but not upon the starting value of x. This establishes an interesting bijection between a subset of infinite bips and the set of non-negative real numbers.

Among potential applications of bips is a qualitatively novel real numbers representation which is also briefly discussed.

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I. Introduction

Given a positive real number β and a real number x, one can form expressions such as

(I.1) $\beta\text{^}{+}\beta\text{^}{-}\beta\text{^}{+}\beta\text{^}{+}\beta\text{^}{-}x \equiv \beta\text{^}(+\beta\text{^}(-\beta\text{^}(+\beta\text{^}(+\beta\text{^}(-x)))))$

where ^ stands for the binary power operator ($\beta\text{^}x \equiv \beta^x$) and, as indicated, the order of evaluation is from right to left. Expressions of this kind shall be called *binary iterated-powers* (or *bips*) in base β. The distribution of the "+" and "-" signs within a bip can be encoded by means of binary sequences having "0" and "1" as their elements. If we let "1" correspond to the "+" sign and "0" to the "-" sign, the above example can be associated with the binary sequence b ≡ {10110} and the value of the expression, intended as a function of x, can be written as {10110|x}.

In this paper we investigate the behavior of bips and show that when $e^{-1} < \text{abs}(\ln(\beta)) \leq e$ and the length of the binary sequence extends to infinity, then *every infinite bip converges to a value which is independent of* x. This, among other things, leads to a mapping of the set *B* of infinite binary sequences into the set *R* of non-negative real numbers, extended by the addition of infinity element. We also show that for a subset $B^*\subset B$, *this mapping is a bijection* and thus gives rise to a *representation of real numbers* by infinite bips, encoded by the corresponding binary sequences.

There is a relationship between bips and the functions known as *iterated exponentials* or *hyperpowers* [1], [2], which can be viewed as very special cases of bips. It turns out, however, that the above-mentioned bijection holds only for β values for which the hyperpowers do not converge to a finite limit. In order to underline this diversity and avoid misunderstandings, we will use preferentially the short term *bip* rather than the expanded form *binary iterated power*.

II. Definitions and notation

Let *R* be the set of all non-negative real numbers with the addition of infinity (∞) as a special element. Since *R* is totally ordered (∞ is defined to be greater than all other elements of *R*), one can easily extend to it the concept of *intervals*. The following intervals of *R* shall be used quite often:

$[0,\infty] \equiv U_{0\infty}$, $[0,1] \equiv U_{01}$ and $[1,\infty] \equiv U_{1\infty}$.

We will use two mappings, $\mathbf{E}^+:R\to R$ and $\mathbf{E}^-:R\to R$, defined as

(II.1) $\mathbf{E}_\pm(x) \equiv \beta\text{^}(\pm x) \equiv \exp(\pm\alpha x)$ for any $x\in[0,\infty)$ and, as an extension, $\mathbf{E}_+(\infty) = \infty$, $\mathbf{E}_-(\infty) = 0$.

Here $\alpha = \ln(\beta)$ and β, called the *base*, is a positive real number. The symbol '±' stands for either '+' or '-' and we follow the usual convention that, within a statement, one must choose systematically either the upper symbol or the lower symbol.

Since $\mathbf{E}_\pm$ will be often used iteratively, it is convenient to define $\mathbf{E}_\pm^n:R\to R$ by setting

(II.2) $\mathbf{E}_\pm^0(x) \equiv x$ and $\mathbf{E}_\pm^{n+1}(x) \equiv \mathbf{E}_\pm(\mathbf{E}_\pm^n(x))$ for any n=0,1,2,...

The convention followed throughout this paper is that mappings are set in bold face and their products and powers are defined by *nesting*, while functions are set in normal type and subject to normal arithmetic. For example, when $\mathbf{M}_k:R\to R$ are mappings corresponding to some functions $m_k(x)$, then $\mathbf{M}_1\mathbf{M}_2(x) \equiv \mathbf{M}_1(\mathbf{M}_2(x))$ and $\mathbf{M}_1^2(x) \equiv \mathbf{M}_1(\mathbf{M}_1(x))$, while $m_1^2(x) \equiv m_1(x).m_1(x) = (m_1(x))^2$. Unlike functions, mappings will be routinely applied also to the power sets of their domains.

Closely associated with $\mathbf{E}_\pm$ are their inverse mappings $\mathbf{L}_\pm:R\to R$. In this particular case, $\mathbf{L}_+$ and $\mathbf{L}_-$ can be lumped together by means of a common function $\mathbf{L}:R\to R$, defined as

(II.3) $\mathbf{L}_\pm(x) \equiv \mathbf{L}(x) = \text{abs}(\ln_\beta(x)) = \text{abs}(\ln(x)/\alpha)$ for any $x\in(0,\infty)$ and $\mathbf{L}(0) = \infty$, $\mathbf{L}(\infty) = \infty$.

The powers of $\mathbf{L}$ are

(II.4) $\mathbf{L}^0(x) \equiv x$ and $\mathbf{L}^{n+1}(x) \equiv \mathbf{L}(\mathbf{L}^n(x))$ for any n=0,1,2,...

The next lemma lists some of the properties of the *base mappings* $\mathbf{E}_\pm$. Though the statements are elementary enough to justify skipping their proofs, we list them anyway since they enumerate the requisites for eventual extensions of the theory to other base mappings.

Lemma 1. The *base mappings* $\mathbf{E}_+:R\to R$, $\mathbf{E}_-:R\to R$ and $\mathbf{L}:R\to R$ have the following properties:
(a) $\mathbf{E}_\pm$ are continuous and have a derivative everywhere in the interval $(0,\infty)$.
(b) $\mathbf{E}_+$ is increasing and $\mathbf{E}_-$ is decreasing everywhere in R.
(c) $\mathbf{E}_\pm$ map any (closed/open) interval U of R onto a (closed/open) interval $\mathbf{E}_\pm(U)$.
(d) The mappings $\mathbf{E}_\pm:U\to\mathbf{E}_\pm(U)$ are bijections whose inverse is in both cases the mapping $\mathbf{L}$.
(e) When $\{U_k\}$, k = 0,1,2,... is a countable set of intervals of R then
$\cap_k\mathbf{E}_\pm(U_k) = \mathbf{E}_\pm(\cap_k U_k)$ and $\cup_k\mathbf{E}_\pm(U_k) = \mathbf{E}_\pm(\cup_k U_k)$.
(f) When $U' \subset U$, then $\mathbf{E}_\pm(U') \subset \mathbf{E}_\pm(U)$.
(g) $\mathbf{E}_-(x) \le 1 \le \mathbf{E}_+(x)$ for any $x\in R$, with the equalities occurring only for x=0.
(h) When $x\in(0,\infty)$, then $\mathbf{E}_\pm(x)\in(0,\infty)$.
(i) $\mathbf{E}_+(U_{0\infty}) = U_{1\infty}$, $\mathbf{E}_-(U_{0\infty}) = U_{01}$ and $\mathbf{E}_+(U_{0\infty})\cup\mathbf{E}_-(U_{0\infty}) = U_{0\infty}$.

We shall call **L**-*progeny* of x the infinite sequence of iterated images (*descendants*) of x under **L**:

(II.5) $\mathbf{P}(x) \equiv \{x_0,x_1,x_2,...\}$, where $x_0 \equiv x$ and $x_k = \mathbf{L}(x_{k-1}) = \mathbf{L}^k(x)$ for any k>0.

Denoting as B the set of all *infinite* binary sequences with elements "0" and "1", we define a mapping $\mathbf{B}:R\to B$ as follows: given an $x\in R$, consider its **L**-progeny $\mathbf{P}(x) \equiv \{x_0,x_1,x_2,...\}$ and associate with x the binary sequence

(II.6) $\mathbf{B}(x) \equiv \{b_0b_1b_2...\}$ in which b_k = "1" when $x_k \ge 1$ and b_k = "0" when $x_k < 1$.

For example, in the neperian base β = e, **B**(2) = {1001 0110 0000 1011 0100 1010 1011 0000...} ≡ {960b4ab0...}, **B**(π) = {d2c52092...} and **B**(10^{100}) = {f0445111...} (the standard hexadecimal transliteration of binary quadruplets is used to keep the notation compact).

Denoting as *B* the set of all *infinite* binary sequences of elements "0" and "1" and as B_n the set of *finite* binary sequences of length n, let us define *starter mappings* $\mathbf{S}_n:B\to B_n$ and *trailer mappings* $\mathbf{T}_n:B\to B$, such that
a) the first n elements of b coincide with the elements of $\mathbf{S}_n$(b), and
b) the k-*th* element of $\mathbf{T}_n$(b) is the (k+n)-*th* element of b.

Explicitly, when b ≡ $\{b_0b_1b_2...\}$, then

(II.7a) $\mathbf{S}_n(b) = \{b_0b_1b_2...b_{n-1}\}$,

(II.7b) $\mathbf{T}_n(b) = \{b_nb_{n+1}b_{n+2}...\}$,

(II.7c) $b = \{\mathbf{S}_n(b)\mathbf{T}_n(b)\}$.

The last expression is an example of the *concatenation of binary sequences* which is an intuitive operation of the type b⊕c ≡ {bc}. Notice, however, that the first argument of {bc} must be a finite binary sequence while the second one may be either finite or infinite.

Given a binary sequence b, finite or infinite, we shall call *complementary* to b the sequence b' = **Cpl**(b) in which *all elements* are inverted with respect to the corresponding ones in b (in other words, when b_k = "0" then b'_k = "1" and vice versa). When *only the first element* of b' is inverted and all the others are identical, the sequence b' shall be called the *inverse* of b and denoted as **Inv**(b).

The following integer-valued functions are useful for *finite* binary sequences:
- len(b), the *length* of b, equal to the total number of elements in sequence b.
- zer(b), the *zeroes count function*, equal to the number of "0" elements in the sequence b.

We define also the *signature function* sgn(b) such that sgn(b) = +1 when the first element of b is "1" and sgn(b) = -1 when it is "0" (in this case it makes no difference whether b is finite or infinite).

It is often necessary to handle binary sequences with long segments composed only of "0"s or "1"s. In such cases, we will use an upper index to encode the length of such segments writing, for example, $\{101^60^5\}$ instead of {1011111100000}. The advantage of such a notation is best evidenced by sequences with variable-length segments, such as $\{01^n\}$.

In infinite sequences one might also encounter periodically repeated segments. We shall use the underline to indicate such periods. Thus, for example, $\{\underline{01}\}$ ≡ {010101...}, $\{0\underline{1}\}$ ≡ {0111...}, etc.

As anticipated in the Introduction, given a positive base β and a *finite* binary sequence b ≡ $\{b_0b_1b_2...b_{n-1}\}$, we associate with it the following binary iterated power, or *bip*, which is a function of the real argument x:

(II.8) $\{b|x\}_\alpha = \beta\text{^}\sigma_0\beta\text{^}\sigma_1\beta\text{^}\sigma_2...\ \beta\text{^}\sigma_{n-1}x = \exp(\alpha\sigma_0 \exp(\alpha\sigma_1 \exp(\alpha\sigma_2 ... \exp(\alpha\sigma_{n-1}x))))$,

where σ_k = +1 when b_k ="1" and σ_k = -1 when b_k ="1". The parameter $\alpha = \ln(\beta)$ can assume any real value and shall be used much more often than the base β. When there is no danger of confusion, the explicit indication of the dependence of $\{b|x\}_\alpha$ on α shall be dropped in favor of the simpler notation {b|x}. Some bips formulae get simplified by admitting also a void binary sequence of length 0 and setting {|x} ≡ x.

The special bip $\{1^n|x\}_\alpha$, evaluated for x = 1, is the finite hyperpower function [1] of the argument α or, more conventionally, of $\beta = e^\alpha$:

(II.9) $\Theta_n(\beta) = \beta\text{^}\beta\text{^}...\text{^}\beta \equiv \mathbf{E}_+^n(1) \equiv \{1^n|1\}_\alpha$.

For $n\to\infty$, $\Theta_n(\beta)$ converges to a finite value when $\alpha \in [-e,e^{-1}]$, diverges (i.e., converges to ∞) when $\alpha > e^{-1}$ and oscillates when $\alpha < -e$ (for proofs, see either [1] or the proof of *Theorem I* of this paper).

For any finite binary sequence b, $\{b|x\}$ is a continuous mapping of the real numbers set **R** into *R* which has finite derivatives of any order with respect to both x and α. The derivatives of $\{b|x\}$ with respect to x will play an important role. For brevity, they shall be written either as $\{b|x\}'$ or $\{b|'x\}$. There is a difference between the two notations which emerges when x is replaced by a function f(x) on **R**:
a) $\{b|f(x)\}'$ is the derivative of the composite function, while
b) $\{b|'f(x)\}$ is the derivative of the function $\{b|y\}$ with respect to y, evaluated at the point y=f(x).

Given an *infinite* binary sequence $b\in B$ and a natural n, consider the following mapping $\mathbf{M}_{n,b}:R\to R$

(II.10) $\mathbf{M}_{n,b}(x) = \{\mathbf{S}_n(b)|x\} = \mathbf{E}_0(\mathbf{E}_1(\mathbf{E}_2(...\mathbf{E}_{n-1}(x))...)))$,

where $\mathbf{E}_k \equiv \mathbf{E}_+$ when b_k = "1" and $\mathbf{E}_k \equiv \mathbf{E}_-$ when b_k = "0". Since $\mathbf{M}_{n,b}$ depends only on the first n elements of b, its definition extends trivially to all binary sequences whose length is at least n. The following lemma summarizes some of its properties:

Lemma 2. For any natural n and any $b\in B$
(a) $\mathbf{M}_{n,b}(x)$ is continuous and has a derivative everywhere in the interval $(0,\infty)$.
(b) $\mathbf{M}_{n,b}(x)$ maps any (closed/open) interval *U* of *R* onto a (closed/open) interval $\mathbf{M}_{n,b}(U)$.
(c) The mapping $\mathbf{M}_{n,b}:U\to\mathbf{M}_{n,b}(U)$ is a bijection whose inverse is $\mathbf{L}^n$.
(d) When $\{U_k\}$, k = 0,1,2,... is a countable set of intervals of *R* then
$\cap_k\mathbf{M}_{n,b}(U_k) = \mathbf{M}_{n,b}(\cap_k U_k)$ and $\cup_k\mathbf{M}_{n,b}(U_k) = \mathbf{M}_{n,b}(\cup_k U_k)$.
(e) When *U'* is a *proper* subinterval of *U* then $\mathbf{M}_{n,b}(U')$ is a *proper* subinterval of $\mathbf{M}_{n,b}(U)$.
(f) $\mathbf{M}_{n,b}(x)$ is increasing when $zer(\mathbf{S}_n(b))$ is even, and decreasing when $zer(\mathbf{S}_n(b))$ is odd.
(g) When $x\in(0,\infty)$, the first n elements of the binary sequence $b' = \mathbf{B}(\mathbf{M}_{n,b}(x))$ coincide with those of b.

Proof.

Statement (a) to (e): $\mathbf{M}_{1,b}$ coincides with $\mathbf{E}_+$ or $\mathbf{E}_-$ (depending upon whether b_0 = "1" or "0", respectively). For n = 1, *statements (a),(b),(c),(d),(e)* coincide with the *statements (a),(c),(d),(e),(f)*, of *Lemma 1, respectively*. It remains to prove that their validity for $\mathbf{M}_{n,b}$ implies that they hold also for $\mathbf{M}_{n+1,b}$. From (II.10) it follows that $\mathbf{M}_{n+1,b}(x) \equiv \mathbf{M}_{n,b}(\mathbf{E}_n(x))$. Since each $\mathbf{E}_n$ is either $\mathbf{E}_+$ or with $\mathbf{E}_-$, however, the validity of *statements (a),(b),(c),(d),(e)* for $\mathbf{M}_{n+1,b}(x)$ follows directly from the assumption and, again, from *Lemma 1*.

Statement (f): For n=1, this statement coincides with that of *Lemma 1b*. Assume now that it holds for $\mathbf{M}_{n,b}(x)$. We distinguish two mutually exclusive cases. Case i) When b_n = "1", $\mathbf{M}_{n+1,b}(x) \equiv \mathbf{M}_{n,b}(\mathbf{E}_+(x))$. Since $\mathbf{E}_+(x)$ is an increasing function of x, $\mathbf{M}_{n,b}(\mathbf{E}_+(x))$ is either increasing or decreasing, depending upon the behavior of $\mathbf{M}_{n,b}(x)$ which, by assumption, depends upon the parity of $zer(\mathbf{S}_n(b))$. Considering that, in this case, $zer(\mathbf{S}_{n+1}(b)) = zer(\mathbf{S}_n(b))$, if follows that *Statement (f)* holds also for $\mathbf{M}_{n+1,b}(x)$. Case ii) When b_n = "0", $\mathbf{M}_{n+1,b}(x) \equiv \mathbf{M}_{n,b}(\mathbf{E}_-(x))$ and, since $\mathbf{E}_-(x)$ is a decreasing function of x, the behavior of $\mathbf{M}_{n+1,b}(x)$ is opposite to that of $\mathbf{M}_{n,b}(x)$. In this case, however, $zer(\mathbf{S}_{n+1}(b)) = zer(\mathbf{S}_n(b))+1$ so that the parity of $zer(\mathbf{S}_{n+1}(b))$ is opposite to that of $zer(\mathbf{S}_n(b))$ and *Statement (f)* again holds also for $\mathbf{M}_{n+1,b}(x)$.

To prove *Statement (g)* consider that, according to (II.5) and (II.11), the k-*th* **L**-descendant of $\mathbf{M}_{n,b}(x)$ is

(II.11) $x_k \equiv \mathbf{L}^k(\mathbf{M}_{n,b}(x)) = \mathbf{E}_k\mathbf{E}_{k+1}...\mathbf{E}_{n-1}(x)$ for any $k < n$.

Applying iteratively *Lemma 1h* we see that $x \in (0,\infty)$ implies $\zeta = \mathbf{E}_{k+1}...\mathbf{E}_{n-1}(x) \in (0,\infty)$. When b_k = "1", we have $\mathbf{E}_k \equiv \mathbf{E}_+$ and therefore $x_k = \mathbf{E}_k(\zeta) > 1$ (*Lemma 1g*) which, by definition, gives b'_k = "1". When b_k = "0", we have $\mathbf{E}_k \equiv \mathbf{E}_-$ so that $x_k = \mathbf{E}_k(\zeta) <1$ which gives b'_k = "0". Consequently, $b'_k = b_k$ for every $k < n$.

We will be particularly interested in the image under $\mathbf{M}_{n,b}$ of the interval $U_{0\infty}$. The closed intervals

(II.12) $U_n(b) = \mathbf{M}_{n,b}(U_{0\infty})$

satisfy the following lemma.

Lemma 3: Given any binary sequence b and any natural n, $U_{n+1}(b) \subset U_n(b)$.

Proof (for an illustrative example, see Table I): Consider again (II.10). The function $\mathbf{E}_n(x)$ maps the interval $U_{0\infty}$ onto the interval $[1,\infty]$ when b_n = "1" or onto [0,1] when b_n = "1". In both cases, therefore, $\mathbf{E}_n(U_{0\infty})$ is a *proper* subinterval of *R* so that, by *Lemma 2e*, $U_{n+1}(b) \equiv \mathbf{M}_{n+1,b}(U_{0\infty}) = \mathbf{M}_{n,b}(\mathbf{E}_n(U_{0\infty})) \subset \mathbf{M}_{n,b}(U_{0\infty}) \equiv U_n(b)$.

The intersection of all the nested intervals $U_n(b)$ shall be denoted as

(II.13) $U(b) = \cap_n U_n(b) = \cap_n \mathbf{M}_{n,b}(U_{0\infty})$.

Since the $U_n(b)$, n = 1,2,3,...form a sequence of closed, nested intervals, $U(b)$ can be either a proper interval or a degenerate one containing a single element of *R*. *Theorem I* shall show that, with the chosen base mappings and for a particular range of bases β, $U(b)$ always belongs to the second category, regardless of the choice of b.

Table I. First few intervals U_n(b) for the binary sequence B(2)

evaluated in the neperian base β=e.

n	inf U_n	sup U_n	$\rho(U_n)$
1	1	∞	∞
2	1	2.71828183	1.71828183
3	1.44466786	2.71828183	1.27361397
4	1.99810779	2.71828183	0.72017404
5	1.99810779	2.55012420	0.55202164
6	1.99810779	2.20321879	0.20511100
7	1.99810779	2.03235896	0.03425117
8	1.99810792	2.03235896	0.03425104
9	1.99810792	2.00546799	0.00736007
10	1.99842671	2.00546799	0.00704128
...	...	...	...
32	1.99999999	2.00000000	5.12...e-9

III. Some properties of finite bips and of their derivatives

Some simple properties of bips are evident directly from the definitions and do not need a particular proof. For example, one can easily verify the following *concatenation rules* applicable to any two *finite* binary sequences b and c (see the discussion of the concatenation of binary sequences following II.7)

(III.1) $\{bc|x\} = \{b|\{c|x\}\}$,

$sgn(bc) = sgn(b)$, $zer(bc) = zer(b)+zer(c)$, $len(bc) = len(b)+len(c)$.

The identity $\exp(\alpha x)\exp(-\alpha x) = 1$ amounts to $\{0|x\}\{1|x\} = 1$. More generally, given any binary sequence b and considering the definition of its inverse sequence **Inv**(b), the identity yields the *inversion rule*

(III.2) $\{b|x\}\{\mathbf{Inv}(b)|x\} = 1$,

associated with the obvious auxiliary identities

$sgn(\mathbf{Inv}(b)) = -sgn(b)$, $zer(\mathbf{Inv}(b)) = zer(b) + (1-sgn(b))/2$, $len(\mathbf{Inv}(b)) = len(b)$.

Another elementary property regards the complementary sequence **Cpl**(b). From (II.8) it is evident that complementing the binary sequence in $\{b|x\}_\alpha$ is equivalent to inverting the sign of α (which, in turn, is the same as replacing the base β by $1/\beta$). We thus obtain the *complement rule*

(III.3) $\{b|x\}_\alpha = \{\mathbf{Cpl}(b)|x\}_{-\alpha}$

associated with the auxiliary identities

$$\mathrm{sgn}(\mathbf{Cpl}(b)) = -\mathrm{sgn}(b),\ \mathrm{zer}(\mathbf{Cpl}(b)) = \mathrm{len}(b)-\mathrm{zer}(b),\ \mathrm{len}(\mathbf{Cpl}(b)) = \mathrm{len}(b).$$

The following elementary identities, valid for any real x and y, shall be of some interest. The extended forms in the right column are obtained from those in the left column by replacing x with {b|x} and applying the concatenation rule (III.1). Further extensions can be obtained by replacing y with {c|y}.

(III.4)
$$\begin{array}{ll}
\{0|x\}\{0|y\} = \{0|x+y\}, & \{0b|x\}\{0|y\} = \{0|\{b|x\}+y\}, \\
\{1|x\}\{1|y\} = \{1|x+y\}, & \{1b|x\}\{1|y\} = \{1|\{b|x\}+y\}, \\
\{0|x\}\{1|y\} = \{1|y-x\} = \{0|x-y\}, & \{0b|x\}\{1|y\} = \{0|\{b|x\}-y\} = \{1|y-\{b|x\}\} \\
\{1|x\}\{0|y\} = \{0|y-x\} = \{1|x-y\}, & \{1b|x\}\{0|y\} = \{1|\{b|x\}-y\} = \{0|y-\{b|x\}\}
\end{array}$$

We shall often need explicit bounds on bips. The simplest ones are easily derived from the explicit forms of the bip functions. The following relations list all possibilities for bips of lengths 1 and 2, assuming $\alpha>0$ and $x\in U_{0\infty}$.

(III.5)
$$0 \le \{0|x\} \le 1 \le \{1|x\} \le \infty,$$
$$0 \le \{01|x\} \le e^{-\alpha} \le \{00|x\} \le 1 \le \{10|x\} \le e^{\alpha} \le \{11|x\} \le \infty,$$

Again, additional relations can be obtained by replacing x with {b|x} and applying the concatenation rule.

The following lemma places more sophisticated upper/lower bounds on products of selected finite bips. Though many such bounds can be deduced, we list only those needed in the rest of the paper.

Lemma 4. Let $b \equiv \{b_0b_1b_2...b_n\}$ be a *finite* binary sequence and {b|x} the respective bip function evaluated in base $\beta = \exp(\alpha)$, with $\alpha > 0$ and $x\in U_{0\infty}$. Then the following statements hold:

(a) For any real $r\ge0$ and $s\ge0$, $\alpha^r\{0|x\}x^s \le \mathrm{Min}[(s/e)\alpha^{r-s}, (r/e)^r x^{s-r}]$.
 Corollary 1: $\alpha^r\{0b|x\}\{b|x\}^s \le \mathrm{Min}[(s/e)^s\alpha^{r-s}, (r/e)^r \{b|x\}^{s-r}]$.
 Corollary 2: $\alpha^r\{0b|x\}\{b|x\}^r \le (r/e)^r$.
 Corollary 3: When $0 \le s \le r$, $\alpha^r\{01b|x\}\{1b|x\}^s \le (r/e)^r$.

(b) For any real $r\ge1$ and $s\ge0$, $\alpha^{r+s}\{01|x\}\{1|x\}^{r-1}x^s \le (s/e)^s(r/e)^r$.
 Corollary 1: $\alpha^{r+s}\{01b|x\}\{1b|x\}^{r-1}\{b|x\}^s \le (s/e)^s(r/e)^r$.
 Corollary 2: $\alpha^3 \{01b|x\}\{1b|x\}\{b|x\} \le 4/e^3$.
 Corollary 3: $\alpha^4 \{01b|x\}\{1b|x\}\{b|x\}^2 \le 16/e^4$.

(c) $\alpha^3\{001|x\}\{01|x\}\{1|x\} \le 27/e^3$.
 Corollary 1: $\alpha^3\{001b|x\}\{01b|x\}\{1b|x\} \le 27/e^3$.

(d) $\{1|x\} \ge e\alpha x$.
 Corollary 1: $\{1b|x\} \ge e\alpha\{b|x\}$.

Proofs:

Statement (a): For a fixed α and variable x, the function $\alpha^r x^s\exp(-\alpha x)$ has a maximum of $(s/e)^s\alpha^{r-s}$, while for a fixed x and variable α, its maximum is $(r/e)^r x^{s-r}$. Hence the result. *Corollary 1* follows replacing x by {b|x}. *Corollary 2* follows from *Corollary 1* by setting s = r and considering just the first Min argument. *Corollary 3* follows from *Corollary 1*, using the second Min argument and noticing that $\{1b|x\}^q \le 1$ for any non-positive q.

Statement (b): Use the identity {1|x}{0|x}=1 and rewrite the left-hand-side side of the inequality as $[\alpha^r\{01|x\}\{1|x\}^r][\alpha^s\{0|x\}x^s]$. It is then sufficient to apply *Statement (a)*, using the first Min argument. *Corollary 1* follows replacing x by {b|x}, *Corollary 2* by setting r = 2, s = 1 and Corollary 3 by setting r = s= 2.

Statement (c): When $\alpha \le 1$, rewrite the expression as $[\alpha^2\{001|x\}].[\alpha\{01|x\}\{1|x\}]$. Since the first factor is smaller than 1 and, by *Statement (a), Corollary 2*, $[\alpha\{01|x\}\{1|x\}] \le 1/e$, one obtains the upper bound of 1/e which is smaller than $27/e^3$. When $\alpha \ge 1$, use (III.4) and rewrite the expression as $\alpha^3\{0|y\}$, where y = {01|x}+{1|x}-x. Given the assumption, {1|x}-x = exp(αx) - x ≥ 1. Considering that {01|x} > 0, we have y>1 and, since {0|y} is a decreasing function, $\{0|y\} \le \{0|1\} = e^{-\alpha}$. Hence $\{001|x\}' \le \alpha^3e^{-\alpha}$ which can not exceed the maximum of $27/e^3$. *Corollary 1* follows replacing x by {b|x}.

Statement (d): The inequality $e^x \ge ex$ gives $\{1|x\} \equiv e^{\alpha x} \ge e\alpha x$. For the Corollary, replace x by {b|x}.

We shall now turn our attention to the derivatives of {b|x} with respect to x. Again, a number of properties follows directly from definitions. Thus, considering that $\{b|x\} = \exp(\alpha\ \mathrm{sgn}(b)\{\mathbf{T}_1(b)|x\})$, one obtains

(III.6) $\{b|x\}' = \alpha\, \mathrm{sgn}(b)\, \{b|x\}\{\mathbf{T}_1(b)|x\}'$

and, through iteration,

(III.7) $\{b|x\}' = \alpha^{len(b)}\ (-1)^{zer(b)}\ \Pi_{k=0,len(b)-1}\ \{\mathbf{T}_k(b)|x\}$,

A further generalization consists in replacing x by a differentiable function f(x):

(III.8) $\{b|f(x)\}' = \{b|'f(x)\}\ f'(x) = (-1)^{zer(b)}\ \alpha^{len(b)}\ [\ \Pi_{k=0,len(b)-1}\ \{\mathbf{T}_k(b)|f(x)\}]\ f'(x)$.

In particular, when $c\in B$ and f(x) = {c|x}, we have (applying the concatenation rule III.1)

(III.9) $\{bc|x\}' \equiv \{b|\{c|x\}\}' = \{b|'\{c|x\}\}\{c|x\}'$

These identities considerably simplify computer evaluation of bips and their derivatives. As an example, consider {01011|x} = {0|{1|{0|{1|{1|x}}}}} whose evaluation requires five exponentiations generating the values {1|x}, {11|x}, {011|x}, {1011|x}, and {01011|x}. To evaluate the derivative {01011|x}', one expands is as $\alpha^5\{01011|x\}\{1011|x\}\{011|x\}\{11|x\}\{1|x\}$ which requires, apart from the factor α^5, the evaluation of the same five values which now appear as factors in the product.

A recursion formula for the derivative of {b|x} with respect to α, analogous to (III.6), is also easily obtained by explicit derivation:

(III.10) $d\{b|x\}_\alpha/d\alpha \equiv \{b|x\}^{\sim} = \mathrm{sgn}(b)\{b|x\}[\{\mathbf{T}_1(b)|x\}+\alpha\{\mathbf{T}_1(b)|x\}^{\sim}]$.

The evaluation of $\{b|x\}^{\sim}$ is again relatively simple and efficient. For example, the evaluation of $\{01011|x\}^{\sim}$ requires the same five exponentiations plus 10 products and 5 additions.

Applying the differentiation rules for nested functions, one obtains

(III.11) $\{b|f(\alpha,x)\}^{\sim} = \{b|^{\sim}f(\alpha,x)\}\ (df(\alpha,x)/d\alpha)$,

which, for concatenated sequences, gives

(III.12) $\{bc|x\}^{\sim} = \{b|^{\sim}\{c|x\}\}\ \{c|x\}^{\sim}$.

The last identity makes it possible to take advantage of efficient sequence-splitting strategies. These do not lead to any appreciable saving in the case of derivatives with respect to x but, given the increased complexity of (III.11), they do save some of the extra products required to evaluate the derivatives with respect to α.

The next lemma places upper bounds on the absolute values of the derivatives of selected types of bips. Again, though many such bounds can be derived; we list only those which will be used in the rest of this text. Notice also that, according to (III.7), the derivative $\{b|x\}'$ is positive when zer(b) is even and negative when it is odd. A minus sign has been used whenever required in order to deal always with positive quantities.

Lemma 5. Let $b \equiv \{b_0 b_1 b_2 ... b_n\}$ be a *finite* binary sequence and $\{b|x\}$ the respective bip function in base $\beta = e^\alpha$, with the values α and x subject to the conditions $\alpha > 0$ and $x \in [0,\infty)$, unless specified otherwise. In addition, let n, m, k and l be any naturals, unless specified otherwise. Then the following statements hold:

(a) $\{00|x\}' \le \alpha/e$.

Corollary: When $\alpha \le e$ and $k > 0$, then $\{0^{2k}|x\}' \le 1$.

(b) $-\{01|x\}' \le 4/e^2$.

(c) When $\alpha \le 1/e$ and $x \in [0,e]$, then $-\{01^n|x\}' \le 1/e^2$.

(d) When $\alpha \ge 4/e^2$, then $-\{01^n|x\}' \le 4/e^2$. When $\alpha \ge 1$, this sharpens to $-\{01^n|x\}' \le (4/e^2)^n$.

(e) When $1/e \le \alpha \le 4/e^2$ and $x \in U_{01}$, then $\{01^n|x\}' < 0.1986$ (numeric estimate) $< 4/e^2$.

(f) When $x \in U_{01}$, then $-\{01^n|x\}' \le 4/e^2$.

(g) When $x \in U_{01}$ and $\alpha \le e$, then $-\{01^n 0^{2k}|x\}' \le 4/e^2$.

(h) $\{001^n|x\}' \le -\alpha \{01^n|x\}'$.

(i) When $x \in U_{01}$, then $\{001^n|x\}' \le 4\alpha/e^2$.

In particular, when $x \in U_{01}$ and $\alpha \le 1$, then $\{001^n|x\}' \le 4/e^2$.

(j) When $\alpha \ge 1$ and $n > 1$, then $\{001^n|x\}' < 16\alpha/e^4$.

In particular, when $1 \le \alpha \le e$, then $\{001^n|x\}' \le 16/e^3$.

(k) When $x \in U_{01}$, $\alpha \le e$ and $n > 1$, then $\{001^n 0^{2k}|x\}' \le 16/e^3$.

(l) $\{001|x\}' < 27/e^3$.

(m) When $\alpha \le e$, then $\{0010^{2k}|x\}' < 27/e^3$.

(n) When $x \in U_{01}$ and $\alpha \le e$ then $-\{0010^{2k}01^n0^{2l}|x\}' \le 108/e^5$.

(o) When $\alpha \le e$, then $-\{0010^{2k+1}|x\}' \le 4/e^2$.

(p) When $x \in U_{01}$ and $\alpha \le e$, then $\{0010^{2k}001^n0^{2l}|x\}' \le 16/e^4$.

Proofs:

Statement (a): By (III.7), $\{00|x\}' = \alpha^2\{00|x\}\{0|x\}$. The inequality is obtained from *Lemma 4a, Corollary 2* for r = 1. The *Corollary* holds for k=1. For k > 1, use (III.9) to write $\{0^{2k+2}|x\}' = \{0^{2k}|'\{00|x\}\}\{00|x\}' = \{0^{2k}|'\xi\}$ with $\xi = \{00|x\}$. When $\alpha \le e$ we have $\xi \le 1$ so that $\{0^{2k+2}|x\}' \le \{0^{2k}|'\xi\}$ and the statement follows by induction.

Statement (b): By (III.7), $-\{01|x\}' = \alpha^2\{01|x\}\{1|x\}$. Now apply *Lemma 4a, Corollary 3* (r=2, s=1).

Statement (c): From (III.6) and (III.9) we have $\{01^{n+1}|x\}' = \{01^n|\{1|x\}\}' = \alpha\ \{01^n|'\{1|x\}\}\{1|x\}$. This points out the special values of $\alpha = 1/e$ and x = e for which $\alpha\{1|x\} = 1$ and the derivatives are the same for all n (see Figure 1b). When $\alpha \le 1/e$ and $x \in [0,e]$, we have $\alpha x \le 1$ so that $\alpha\{1|x\} = \alpha.\exp(\alpha x) \le \alpha e \le 1$ and $|\{01^{n+1}|x\}'| \le |\{01^n|'\xi\}|$ with $\xi = \{1|x\} \le e$. Since $\xi \in [0,e]$, the process can be iterated until $|\{01^{n+1}|x\}'| \le |\{01|'\eta\}|$, with $\eta \in [0,e]$. By *Lemma 4a, Corollary 2*, the function $|\{01|x\}'| = \alpha^2\{01|x\}\{1|x\}$ can not exceed α/e which, given the assumption, is bounded by $1/e^2$.

Statement (d): Applying twice over (III.6), one obtains

$$\{01^{n+1}|x\}' = \alpha^2\ \{01^{n+1}|x\}\{1^{n+1}|x\}' = \alpha^2\ \{01^{n+1}|x\}\{1^{n+1}|x\}\{1^n|x\}'.$$

Multiplying the right-hand side by the unity $\{1^{n+1}|x\}\{01^n|x\}$ (see III.2), we obtain

$$\{01^{n+1}|x\}' = \alpha^2\ \{01^{n+1}|x\}\{1^{n+1}|x\}^2\ \{01^n|x\}\{1^n|x\}' = \alpha\ \{01|\xi\}\{1|\xi\}^2\ \{01^n|x\}', \text{ where } \xi = \{1^n|x\}.$$

Applying *Lemma 4a, Corollary 2* with r = 2, this yields

$$|\{01^{n+1}|x\}'| \le (4/\alpha e^2)\ |\{01^n|x\}'|.$$

For $\alpha \geq 4/e^2$, the inequality gives $|\{01^{n+1}|x\}'| \leq |\{01^n|x\}'|$ which iterates to $|\{01^{n+1}|x\}'| \leq |\{01|x\}'|$. For $\alpha \geq 1$ the inequality sharpens to $|\{01^{n+1}|x\}'| \leq (4/e^2)|\{01^n|x\}'|$ which iterates to $|\{01^{n+1}|x\}'| \leq (4/e^2)^n|\{01|x\}'|$. The rest follows from *Statement (b)*.

Statement (e): Within the interval $1/e< \alpha <4/e^2$, the functions $\{01^n|x\}'$ defy explicit analysis. The graphs shown in Figures 1c) to 1f) illustrate the behavior of $|\{01^n|x\}'|$ for n = 1 to 60 and for several values of α distributed across and close to this interval (the shown graphs were selected from a much more detailed set). In order to clarify the overall behavior of the functions, they are plotted over a range of x-values which is considerably wider than the interval U_{01} of interest. For $x \in U_{01}$, the absolute values of the derivatives remain abundantly below $4/e^2$. Numeric search for the maximum with respect to n, α and x shows beyond any doubt that for $\alpha \in [1/e,4/e^2]$ and $x \in [0,1]$, the functions never exceed 0.1986... (most probably the value of $-\{01|x\}'$, evaluated for $\alpha = 4/e^2$ and x=1). This is abundantly smaller than $4/e^2$ (equal to about 0.5413...). Figure 1i provides an additional and very convincing (though not essential) illustration of this fact.

Statement (f) is a logical combination of *statements (c)*, *(d)* and *(e)*.

Statement (g):.When k=0, *Statement (f)* is directly applicable. For $k \geq 1$, use (III.9) to write $\{01^n0^{2k}|x\}' = \{01^n|'\{0^{2k}|x\}\}\{0^{2k}|x\}'$ and apply the *Corollary* of *Statement (a)* to the second term and *Statement (f)* to the first one (notice that $\{0^{2k}|x\} \in U_{01}$ for any x so that, when $k \geq 1$, the condition $x \in U_{01}$ may be dropped).

Statement (h): Use (III.9) to write $\{001^n|x\}' = -\alpha\, \{001^n|x\}\{01^n|x\}'$ and consider that $\{001^n|x\} \leq 1$ for any x.

Statement (i) is a combination of statements (f) and (h).

Statement (j): According to (e), when $\alpha \geq 1$ and $n > 1$, $-\{01^n|x\}' \leq (4/e^2)^n \leq 16/e^4$.
Applying this to (i), we have $\{001^n|x\}' \leq 16\alpha/e^4$.

Statement (k): Combining *Statement (i)* and *Statement (j)*, we deduce that when $x \in U_{01}$, $\alpha \leq e$ and $n > 1$, then $\{001^n|x\}' < \max(16/e^3,4/e^2) = 16/e^3$. For $k > 1$, use (III.9) to write $\{001^n0^{2k}|x\}' = \{001^n|'\{0^{2k}|x\}\}\{0^{2k}|x\}'$ and apply the result for k = 0 to the first term and the *Corollary* of *Statement (a)* to the second one.

Statement (l): Since $\{001|x\}' = \alpha^3\{001|x\}\{01|x\}\{1|x\}$, apply *Lemma 4c*.

Statement (m): When k=0, *Statement (l) is directly applicable*. For k>0, use (III.9) to write $\{0010^{2k}|x\}' = \{001|'\{0^{2k}|x\}\}\{0^{2k}|x\}'$ and apply *Statement (l)* to the first factor and the *Corollary* of *Statement (a)* to the second one.

Statement (n): By (III.9), $\{0010^{2k}01^n0^{2l}|x\}' = \{0010^{2k}|'\{01^n0^{2l}|x\}\}\{01^n0^{2l}|x\}'$. Apply *Statement (m)* to the first factor and *Statement (g)* to the second one.

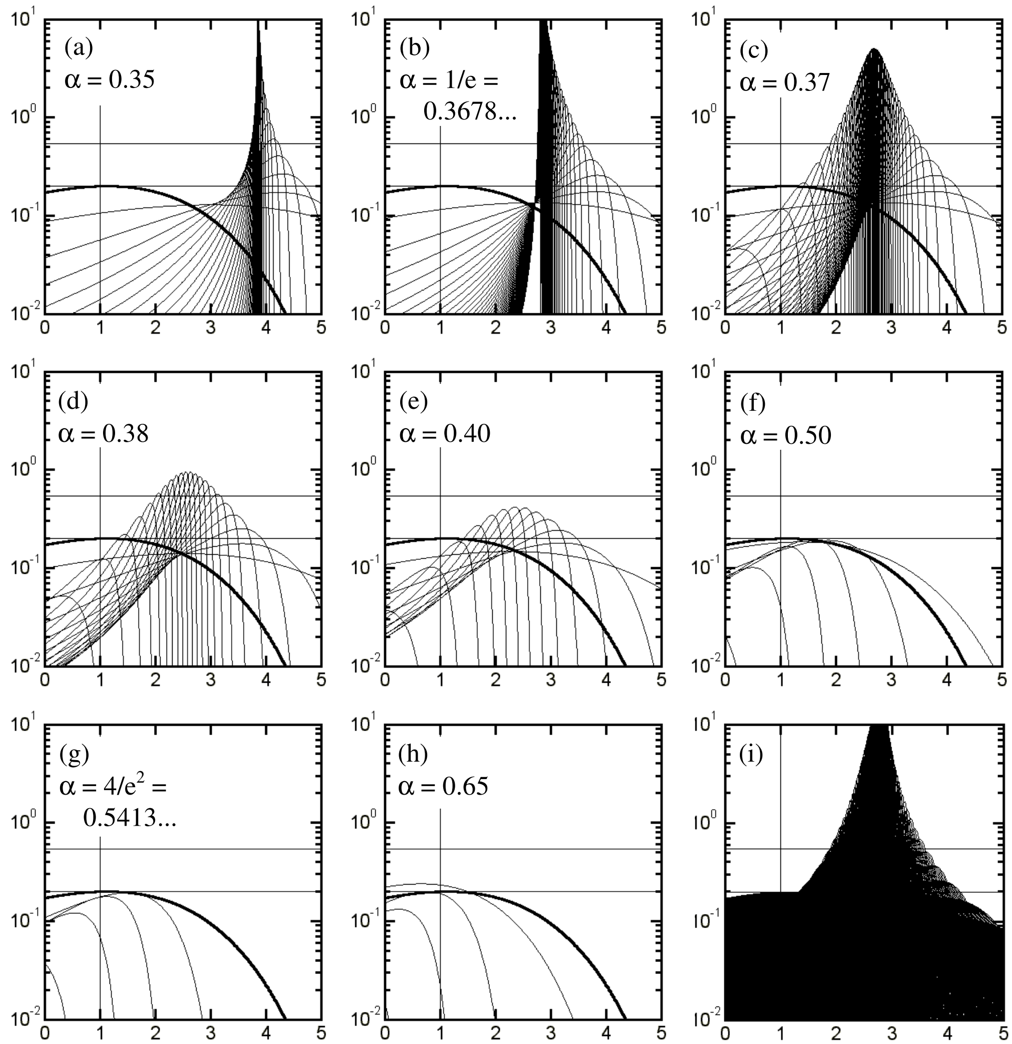
Statement (o): By (III.7), $-\{0010|x\}' = \alpha^4\{0010|x\}\{010|x\}\{10|x\}\{0|x\}$. Considering that $\{0010|x\} \leq 1$ and applying *Lemma 4b, Corollary 2*, to the last three terms, one obtains $-\{0010|x\}' \leq 4\alpha/e^3 \leq 4/e^2$. For $k \geq 1$, use (III.9) to write $\{0010^{2k+1}|x\}' = \{0010|'\{0^{2k}|x\}\}\{0^{2k}|x\}'$ and apply the result for k=0 to the first term and the *Corollary* of *Statement (a)* to the second one.

Statement (p): Use (III.9) to write $\{0010^{2k}001^n0^{2l}|x\}' = \{0010^{2k+1}|'\{01^n0^{2l}|x\}\}\{01^n0^{2l}|x\}'$ and apply *Statement (o)* to the first term and *Statement (g)* to the second one.

The upper bounds set by *Lemma 5* are often grossly conservative due to the fact that, in general, the distinct factors in a derivative expression do not attain their individual bounds simultaneously. We shall see, however, the only thing that really matters for the purposes of this paper is whether the absolute value of a derivative is smaller than 1.

To a purist it will appear unfortunate that the proof of *Lemma 4, Statement (e)* relies on numeric analysis. Such a criticism may be countered by pointing out that, in cases where explicit analytical solution is not available, renouncing to advance a field only because numerical analysis looks less respectable would not be wise - especially since the convincing numeric evidence is likely to stimulate others in a quest for a more 'noble' proof. In the age of advancing experimental mathematics [3], a cautious use of numeric results in proofs is becoming an accepted fact.

Figure 1. Functions -{01^n|x}'



Each of the graphs (a)-(h) plots the functions -{01^n|x}' for a value of α and for n=1-60 (thin, with the rightmost maximum corresponding to n=1). The bold line present in all graphs corresponds to -{01|x}' for $\alpha = 4/e^2$. Notice the extreme sensitivity of -{01^n|x}' to variations of α in the region around 1/e.

The condition we want to verify is that, within the interval $x\in[0,1]$, all the functions are bounded by the value $4/e^2 = 0.5413...$ (upper horizontal line). This has been proved analytically for $\alpha \le 1/e$ and $\alpha \ge 4/e^2$ but needs to be demonstrated for $\alpha\in(1/e, 4/e^2)$. Numerical analysis shows that within the 2D interval of $x\in[0,1]$ and $\alpha\in(1/e, 4/e^2)$, none of the functions exceeds 0.1986.... (lower horizontal line) which seems to coincide with -{01|x}', evaluated at x = 1 and $\alpha = 4/e^2$. This is illustrated in graph (i) which contains the 20000 functions -{01^n|x}' computed for n = 1-100 and for 200 α-values evenly distributed within the interval $[1/e,4/e^2]$.

The remarkable behavior of -$\{01^n|x\}'$ for large n in the immediate vicinity of $\alpha = 1/e$ (Figures 1a,b,c) is quite complicated but fortunately occurs outside the x-values range which interests us here. Its nature is easy to comprehend since, when $\alpha = 1/e$, then $\lim_{n\to\infty}\{1^n|x\}$ is equal to e for $x \le e$ and to infinity for $x > e$ (see Figure 2) thus forming, in the limit, a sharp discontinuity at x = e.

IV. The convergence theorem

The following three lemmas are the last ones needed for the proof of *Theorem I*.

Lemma 6. Let b and b' be two infinite binary sequences such that, for some natural n, $b' = \mathbf{T}_n(b)$ and $U(b')$ contains a single element r' of R. Then $U(b)$ also contains just one element r of R and $r = \mathbf{M}_{n,b}(r') \equiv \{\mathbf{S}_n(b)|r'\}$.

Proof. From the premises we have $U_{n+k}(b) \equiv \mathbf{M}_{n+k,b}(U_{0\infty}) = \mathbf{M}_{n,b}(\mathbf{M}_{k,b'}(U_{0\infty})) = \mathbf{M}_{n,b}(U_k(b'))$ for any $k \ge 0$. Since the consecutive intervals $U_i(b)$ are nested, $U(b) = \cap_i U_i(b) = \cap_k U_{n+k}(b) \Rightarrow U(b) = \cap_k \mathbf{M}_{n,b}(U_k(b'))$. Applying *Lemma 2d*, this gives $U(b) = \mathbf{M}_{n,b}(\cap_k U_k(b'))$ and since, by assumption, $\cap_k U_k(b')$ contains just the element r', $U(b)$ necessarily also contains just one element, namely $r = \mathbf{M}_{n,b}(r')$.

Lemma 7. Given a binary sequence b, let $\{U_n(b)\}$ be the sequence of intervals defined by (II.12). Assume further that there exists a subsequence $\{U_{k(i)}(b),\ i=1,2,3,...,\ k(i) < k(i+1)\}$ such that $\lim_{i\to\infty} \rho(U_{k(i)}(b)) = 0$. Then also $\lim_{n\to\infty} \rho(U_n(b)) = 0$ and the interval $U(b) = \cap_n U_n(b)$ contains just one element of R.

Proof: Due to the premises, there exists for every n an i_n such that $k(i_n) \le n < k(i_n+1)$ and $\lim_{n\to\infty} k(i_n) = \infty$. By *Lemma 3*, the consecutive intervals $U_n(b)$ are nested, so that $\rho(U_n(b)) \le \rho(U_{k(i_n)}(b))$. Hence $\lim_{n\to\infty} \rho(U_n(b)) \le \lim_{n\to\infty} \rho(U_{k(i_n)}(b)) = \lim_{i\to\infty} \rho(U_{k(i)}(b)) = 0$ and, since $\rho(U) \ge 0$ for any interval U, $\lim_{n\to\infty} \rho(U_n(b)) = 0$.

Lemma 8. Let f(x) be a function on R which is monotonous and differentiable everywhere in a finite interval (a,b). Assume further that there exists a value $\eta > 0$ such that $|(f'(x))| \le \eta$ for every $x \in (a,b)$. Then f(x) maps (a,b) onto an interval (a',b') whose size satisfies the inequality $\rho((a',b')) \le \eta\rho((a,b)) = \eta(b-a)$.

Proof: Since f(x) is continuous and monotonous in the interval (a,b), it maps it again onto an interval and the size of the image interval is $\rho(f((a,b))) = |f(b)-f(a)|$. Given the existence of the derivative in (a,b), one has

$$\rho(f(a,b)) = |f(b) - f(a)| = |\int_a^b f'(x)dx| \le \int_a^b |f'(x)|dx \le \eta(b-a).$$

> ***Theorem I.***
> Given an *infinite* binary sequence $b \in B$, denote as $U(b)$ the interval of R defined by (II.13) in base $\beta = e^{\alpha}$.
> Denote further as C_0 the set $[-e,-e^{-1}) \cup [0] \cup (e^{-1},e]$. Then
> (1) When $\alpha \in C_0$ then, for any $b \in B$, $U(b)$ contains a single element of R.
> (2) When $\alpha \notin C_0$ then there exists a sequence $b^{\#} \in B$ for which $U(b)$ contains more than one element of R.

Proof:

a) When $\alpha = 0$, we have $\{b_n|x\} = 1$ for any $b_n \in B_n$. The intervals $U_n(b)$ are therefore all degenerate and contain just the element 1. Consequently, so does $U(b)$ and *Statement (1)* is satisfied.

b) According to (III.3), inverting the sign of α is equivalent to complementing the sequence b. Since every sequence $b \in B$ has in B a complement **Cpl**(b), whenever either of the two *Statements* holds for a particular value of α, it holds also for $-\alpha$. It is therefore sufficient to prove the theorem for positive values of α.

c) Consider the sequence $b \equiv \{\underline{1}\}$ for which $U_n(\{\underline{1}\}) = [\{1^n|0\},\infty]$.

When $\alpha > e^{-1}$, we apply *Lemma 4d* which gives $\{1|x\} \geq e\alpha x = x + x\eta$, where $\eta = e\alpha - 1 > 0$. Considering that $\{1^n|x\} \geq 1$, the inequality can used to write $\{1^{n+1}|x\} = \{1|\{1^n|x\}\} \geq \{1^n|x\} + \{1^n|x\}\eta \geq \{1^n|x\} + \eta$. By iteration, therefore, $\{1^{n+1}|x\} \geq \{1|x\} + n\eta$, which proves that $\{1^n|x\}$ diverges to ∞ for any $x \in R$. Consequently, $\lim_{n\to\infty}\{1^n|0\} = \infty$ so that $U(\{\underline{1}\})$ contains just the element $\infty \in R$, in accordance with *Statement (1)*.

When $0 < \alpha \leq e^{-1}$, the increasing function $\{1|x\} = \exp(\alpha x)$ intersects the line $y = x$ in two points $r_1 < e < r_2$ which are the two roots of the equation $\{1|x\} = x$ (see Fig.2a). The derivative $\{1|x\}'$ is itself an increasing function smaller than 1 at $x = r_1$ and larger than 1 at $x = r_2$. Consequently, r_1 is an attractor of $\{1|x\}$ while r_2 is a repulsor. It is easy to show by textbook means that repetitive applications of $\{1|x\}$, i.e., the functions $\{1^n|x\}$, converge to r_1 for any $x < r_2$, diverge to ∞ for $x > r_2$ and remain invariant for $x = r_2$. The nested intervals $U_n(\{\underline{1}\}) \equiv [\{1^n|0\},\infty]$ thus converge to the non-degenerate interval $U(\{\underline{1}\}) = [r_1,\infty]$.

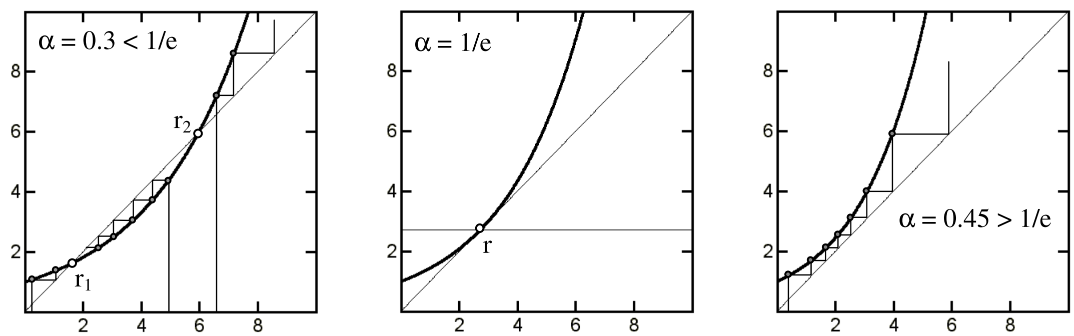


Figure 2. The mapping {1|x}

The graphs are drawn for three different values of α. (a) When $\alpha<1/e$, the equation $\{1|x\} = x$ has two distinct roots r_1 and r_2, the first of which is an attractor and the second a repulsor. The consecutive images of three different x-values under the repeated mappings $\{1^n|x\}$ are shown (small dots). Clearly, $\lim_{n\to\infty}\{1^n|x\}=r_1$ for any $x<r_2$ and $\lim_{n\to\infty}\{1^n|x\}=\infty$ for $x>r_2$. (b). (b) For $\alpha=1/e$, the two roots coalesce into $r = e$, but the behavior remains essentially the same. (c) When $\alpha>1/e$, there is no root and $\lim_{n\to\infty}\{1^n|x\}=\infty$ for any x.

It is worth noticing that, while each $U_n(\{\underline{1}\})$ is a proper interval and a continuous image of $U_{n-1}(b)$ under $\{1|x\}$, the intersection $U(\{\underline{1}\})$ of all such intervals contains points which *can not* be written in the form $\lim_{n\to\infty}\{1^n|x\}$ for some $x \in U_{0,\infty}$. The limit $\lim_{n\to\infty}\{1^n|x\}$ in fact exists for every $x \in R$ but it can assume only one of three possible values, namely r_1, r_2 and ∞. If we interpret the definition (II.13) as an intersection of intervals, the result is the interval $[r_1,\infty]$; while if we intend it as an intersection of subsets, the result is a discrete set containing just three elements.

When $\alpha = e^{-1}$, the two roots r_1 and r_2 coalesce into a single one ($r_1 = r_2 = r \equiv e$) which is an attractor for $x < r$ and a repulsor for $x > r$ (Fig.2b). The nested intervals $U_n(\{\underline{1}\}) \equiv [\{1^n|0\},\infty]$ now converge to the interval $U(\{\underline{1}\}) = [r,\infty]$ and the discrete set of the possible values of $\lim_{n\to\infty}\{1^n|x\}$ contains only r and ∞.

What interests us in the present context is that when $0 < \alpha \leq e^{-1}$ then, regardless of the interpretation of $U(b)$, *Statement (2)* is satisfied by identifying $b^\#$ with sequence $\{\underline{1}\}$.

d) Next we shall concentrate on the sequence $\{\underline{0}\}$.

Consider the odd intervals $U_{2n+1}(\{\underline{0}\}) = [\{0^{2n+1}|0\},\{0^{2n+1}|\infty\}]$. Since $\{0^{2n+1}|x\}$ is a decreasing function of x, $U_{2n+1}(\{\underline{0}\}) = [\{0^{2n+1}|\infty\},\{0^{2n+1}|0\}] = [\{0^{2n}|\{0|\infty\}\},\{0^{2n}|\{0|0\}\}] = [\mathbf{Z}^n(0),\mathbf{Z}^n(1)] = \mathbf{Z}^n(U_{01})$, where the iterated mapping $\mathbf{Z}(x) \equiv \{00|x\}$, is a monotonously increasing function which maps U_{01} into itself. By *Lemma 5a*, the derivative of $\mathbf{Z}(x)$ does not exceed α/e. Iterative application of *Lemma 8* therefore yields $\rho([\mathbf{Z}^n(0),\mathbf{Z}^n(1)]) \le (\alpha/e)^n$ so that, when $0 < \alpha < e$, the sizes of the odd intervals $U_{2n+1}(\{\underline{0}\})$ converge to zero. *Lemma 7* extends this to $\lim_{n\to\infty} \rho(U_n(\{\underline{0}\})) = 0$.

When $\alpha = e$, the situation requires a more subtle analysis (Fig.3b). The equation $\mathbf{Z}(x) = x$ has in this case a single root at $r = (1/e)$ and $\mathbf{Z}'(r) = 1$. Since $\mathbf{Z}'(r) < 1$ for any element $x \neq r$ of U_{01}, consecutive applications of $\mathbf{Z}(x)$ to U_{01} produce a series of nested intervals $U_{2n+1}(\{\underline{0}\})$ each of which brackets r. Suppose that the series $\{U_{2n+1}(\{\underline{0}\})\}$ converges to some non-degenerate limit interval U_L. Then the extremes of U_L would have to be two numbers a and b such that $a < r < b$ and $\mathbf{Z}(a) = a$, $\mathbf{Z}(b) = b$, contradicting the fact that r is a *unique* solution of $\mathbf{Z}(x) = x$. The series of intervals therefore converges to the single point r and, again, $\lim_{n\to\infty} \rho(U_n(\{\underline{0}\})) = 0$.

We have thus shown that *Statement (1)* holds for the sequence $\{\underline{0}\}$ and any $0 < \alpha \le e$.

When $\alpha > e$ (Fig.3c), the equation $\mathbf{Z}(x) = x$ has in U_{01} three roots $r_1 < r_2 < r_3$ such that $|\mathbf{Z}'(r_1)| < 1$, $|\mathbf{Z}'(r_2)| > 1$ and $|\mathbf{Z}'(r_3)| < 1$. The external roots r_1 and r_3 are therefore attractors while the central one (coincident with the root of $\{0|x\}=x$) is a repulsor. Repeated applications of $\mathbf{Z}(x)$ map the interval $[r_1,r_3]$ *onto* itself and make any point of U_{01} external to the interval $[r_1,r_3]$ converge towards either r_1 or r_3, whichever is closer. Consequently, $\lim_{n\to\infty} U_n(\{\underline{0}\})$ coincides with the non-degenerate interval $[r_1,r_3]$.

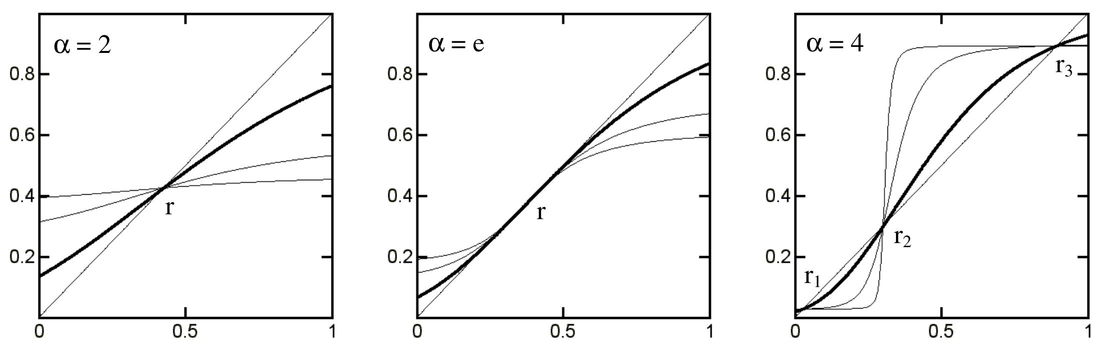


Figure 3. Mappings $\{Z^n|x\}$ in the interval $x \in [0,1]$

The three graphs illustrate the mappings $\mathbf{Z}^n(x) \equiv \{0^{2n}|x\}$ for n = 1 (bold), 4 (intermediate) and 8 for three values of α. (a) When $\alpha < e$, the equation $Z(x)=x$ has only one root r which is an attractor (i.e., repeated mappings $\mathbf{Z}(x)$ map x ever closer to r) and $\lim_{n\to\infty} \mathbf{Z}^n(x) = r$ for any $x \in [0,1]$. (b) When $\alpha = e$, there is still just one root which can be shown to be also an attractor (see text). (c) When $\alpha > e$, however, there are three roots r_1, r_2 and r_3 of which r_1 and r_3 are attractors (derivative $\mathbf{Z}'(x)$ smaller than 1) and r_2 is a repulsor (derivative greater than 1) and we have $\lim_{n\to\infty} \mathbf{Z}^n(x) = r_1$ for $x < r_2$ and $\lim_{n\to\infty} \mathbf{Z}^n(x) = r_3$ for $x > r_2$.

When $\alpha > e$, the sequence $\{0^n|x\}$ is stationary for $x = r_2$. Any other choice of x, however, leads to sequence which has two x-independent accumulation points, r_1 and r_3, and *oscillates* between their neighborhoods. Since this excludes the existence of the limit $\lim_{n\to\infty} \{0^n|x\}$, it follows that when $\alpha > e$, *Statement (2)* can be satisfied by identifying $b^\#$ with the sequence $\{\underline{0}\}$. This, together with the results of Sections (c) and (b) of this proof, concludes the proof of *Statement (2)*.

e) When $\alpha \in (e^{-1},e]$, both sequences $\{\underline{0}\}$ and $\{\underline{1}\}$ satisfy *Statement (1)*. From *Lemma 6* it then follows that *Statement (1)* holds for any sequence $b \in B$ which contains either $\{\underline{0}\}$ or $\{\underline{1}\}$ as a trailer, i.e., one for which there exists a natural n such that either $\mathbf{T}_n(b) = \{\underline{0}\}$ or $\mathbf{T}_n(b) = \{\underline{1}\}$.

f) We shall now analyze the case of $\alpha \in (e^{-1},e]$ and sequences $b \in B$ which start with "0" and *do not* have $\{\underline{0}\}$ or $\{\underline{1}\}$ as a trailer. A basic feature of such sequences is that, given any natural k, there exist natural $n \geq k$ and $m \geq k$ such than b_n="0" and b_m="1". Consequently, each such sequences can be interpreted as an unending concatenation of *finite* fragments belonging to one of the following types (with $n \geq 1$, $k \geq 0$):

type $V_{n,k}$: $\{01^n 0^{2k}\}$,
type $W_{n,k}$: $\{001^n 0^{2k}\}$.

It is evident that the fragmentation is unique (though this is not really essential for the purposes of this proof). For reasons which shall become clear later, we shall not use directly the $W_{1,k}$ fragments but rather the following P and Q fragments with $n \geq 1$ and $k,l \geq 0$:

type $P_{k,n,l}$: $\{001\, 0^{2k}\, 01^n\, 0^{2l}\} \equiv W_{1,k}\, V_{n,l}$,
type $Q_{k,n,l}$: $\{001\, 0^{2k}\, 001^n\, 0^{2l}\} \equiv W_{1,k}\, W_{n,l}$.

This increases the number of fragment types but, since every $W_{1,k}$ fragment must be necessarily followed by one of the two basic types, it does not affect the uniqueness of the fragmentation. As an example, consider the sequence {001 0110000 01111 001100 01...} which leads first to the chain of fragments $\{W_{1,0}V_{2,2}V_{4,0}W_{2,1}...\}$ and, subsequently, $\{P_{0,2,2}V_{4,0}W_{2,1}...\}$.

The i-*th* fragment shall be denoted generically as F_μ, μ=0,1,2,..., and the index of its starting element as k_μ (this implies k_0=0 and $k_\mu < k_{\mu+1}$ for any μ).

Given an integer $\nu \geq 0$, consider the mapping $\mathbf{M}_{j,b}(U_{0\infty})$ for $j = k_\nu$. Since b_j is the starting element of fragment F_ν, it is certainly "0" and since $\{0|x\}$ maps $U_{0\infty}$ onto U_{01}, this implies that $\mathbf{M}_{j,b}(U_{0\infty}) = \mathbf{M}_{j-1,b}(U_{01})$. Considering (II.10) and our definitions of the fragments, we can write $\mathbf{M}_{j-1,b}(x) = \{F_0F_1F_2...F_{\nu-1}|x\} = \mathbf{F}_0\mathbf{F}_1\mathbf{F}_2...\mathbf{F}_{\nu-1}(x)$, where the partial mappings $\mathbf{F}_\mu$ are defined as $\mathbf{F}_\mu(x) \equiv \{F_\mu|x\}$ which amount to

$\mathbf{F}_\mu(x) \equiv \{\mathbf{V}_{n,k}|x\} = \{01^n 0^{2k}|x\}$ when F_μ is of type $V_{n,k}$,
$\mathbf{F}_\mu(x) \equiv \{\mathbf{W}_{n,k}|x\} = \{001^n 0^{2k}|x\}$ when it is of type $W_{n,k}$ (n>1),
$\mathbf{F}_\mu(x) \equiv \{\mathbf{P}_{k,n,l}|x\} = \{001\, 0^{2k}\, 01^n\, 0^{2l}|x\}$ when it is of type $P_{k,n,l}$, and
$\mathbf{F}_\mu(x) \equiv \{\mathbf{Q}_{k,n,l}|x\} = \{001\, 0^{2k}\, 001^n\, 0^{2l}|x\}$ when it is of type $Q_{k,n,l}$.

Combining these facts, we have $U_j(b) \equiv \mathbf{M}_{j,b}(U_{0\infty}) = \mathbf{F}_0\mathbf{F}_1\mathbf{F}_2...\mathbf{F}_{\nu-1}(U_{01})$.

Since nested operators are executed from right to left, the execution of $\mathbf{F}_\mu(x)$ terminates always with the mapping $\mathbf{E}^- \equiv \{0|x\}$. Considering the continuity and monotonicity of the mappings, it follows that, given any interval $U \subseteq U_{01}$, each $\mathbf{F}_\mu$ maps it again onto a sub-interval of U_{01}.

For $x \in U_{01}$, the absolute values of the derivatives of all the mappings $\mathbf{F}_\mu(x)$ are bounded by $\lambda = 16/e^3$, since

$|\{\mathbf{V}_{n,k}|x\}'| = |\{01^n 0^{2k}|x\}'| \leq 4/e^2 < 16/e^3$, according to *Lemma 5g*,
$|\{\mathbf{W}_{n,k}|x\}'| = |\{001^n 0^{2k}|x\}'| \leq 16/e^3$, according to *Lemma 5k*,
$|\{\mathbf{P}_{k,n,l}|x\}'| = |\{001\, 0^{2k}\, 01^n\, 0^{2l}|x\}'|' \leq 108/e^5 < 16/e^3$, according to *Lemma 5n*, and
$|\{\mathbf{Q}_{k,n,l}|x\}'| = |\{001\, 0^{2k}\, 001^n\, 0^{2l}|x\}'| \leq 16/e^4 < 16/e^3$, according to *Lemma 5p*.

According to *Lemma 8* this implies that, when $U \subseteq U_{01}$ and $e^{-1} < \alpha \leq e$, then the size of the image interval $\mathbf{F}_\mu(U)$ satisfies the inequality $\rho(\mathbf{F}_\mu(U)) \leq \lambda\, \rho(U)$. In other words, each of the mappings $\mathbf{F}_\mu$ shrinks the source interval U by at least a pre-determined factor $\lambda < 1$. Returning to the interval $U_j(b)$, it follows that its size has an upper bound given by $\rho(U_{k\nu}(b)) = \rho(\mathbf{F}_0\mathbf{F}_1\mathbf{F}_2...\mathbf{F}_{\nu-1}(U_{01})) \leq \lambda^\nu$ and therefore $\lim_{\nu\to\infty} \rho(U_{k\nu}(b)) = 0$.

We have thus found a subsequence of the intervals U_n(b) which converges to a single point of *R*. It is now sufficient to apply *Lemma 7* to see that the same applies also to the sequence U_n(b), which concludes the proof of the validity of *Statement (1)* for sequences b of the considered type.

g) The remaining case to be analyzed is that of $\alpha\in(e^{-1},e]$ and sequences b∈*B* which start with "1" and do not have {$\underline{0}$} or {$\underline{1}$} as a trailer. When b is such a sequence, there necessarily exists a finite n such that b_n = "0". The trailer T_n(b) is then of the type considered in Section (f) of this proof and thus satisfies *Statement (1)*. Applying *Lemma 6*, it follows that the same holds also for the original sequence b.

h) The combination of Sections (c) through (g) of this proof covers all possible infinite sequences and thus proves the validity of *Statement (1)* for any b∈*B* and $\alpha\in(e^{-1},e]$. To complete the proof, it is sufficient to combine this result with those of Sections (a) and (b).

The reason why we have avoided fragments of the type $W_{1,k}$ is that, by *Lemma 5m*, the lowest upper bound on the derivative of $\mathbf{W}_{1,k}(x)$ is $27/e^3$, which *exceeds* than 1 (there indeed exist values of $x\in U_{01}$ and $\alpha\in(e^{-1},e]$ for which {001|x}' > 1).

The following corollary is an elementary consequence of *Theorem I* and of the basic properties of the real numbers set. We mention it explicitly because its particular wording will become useful later on.

Corollary. Given any infinite binary sequence b∈*B*, a base $\beta=e^\alpha$ such that $\alpha\in C_0$, and any x∈*R*, the sequence {$\mathbf{S}_n$(b)|x} converges to an element r(b, α)∈*R* which is independent of x.

One must stress that when $\alpha\notin C_0$, the sequence {$\mathbf{S}_n$(b)|x} may still converge to a unique element of *R*. The problem is that even if does so, it is not possible to *guarantee* that the limit exists for any choice of b∈*B* and that its value does not depend upon the starting point x. To illustrate the situation, let us consider a few examples.

(a) Sequence b = {$\underline{1}$}. We have seen that when 0< α ≤1/e, the equation {1|x} = x has two roots $r_1(\alpha)$ and $r_2(\alpha)$, $r_1 \le e \le r_2$, and that {$\mathbf{S}_n$(b)|x} ≡ {1^n|x} converges to r_1 for any x < r_2, remains invariant for x = r_2 and diverges (i.e., converges to ∞) for x > r_2. When α is negative, $\mathbf{S}_n$(b) equals $\mathbf{S}_n$(**Cpl**(b)) evaluated for α' = -α. It was shown that the complementary sequence **Cpl**(b) = {$\underline{0}$} satisfies Statement (1) of the Theorem for any α' ≤ e but beyond that point {$\mathbf{S}_n$(**Cpl**(b))|x} starts oscillating between the neighborhoods of two distinct solutions of {00|x}=x. Consequently, {$\mathbf{S}_n$(b)|x} converges for any α ≥ -e and oscillates for α < -e. Moreover, while the convergence of {$\mathbf{S}_n$(b)|x} in *R* is guaranteed for any α ≥ -e, the independence of the limit with respect to the choice of x occurs only when α > 1/e or -e ≤ α ≤ 0.

Notice that within the region $\alpha\in[-e,e^{-1})$ where the limit exists and is *finite*, $r_1(\alpha)$ coincides with the hyperpower function [1] Θ(β) = {$\underline{1}$|1} of $\beta = e^\alpha$.

(b) Sequence b = {$\underline{01}$}. It follows from *Lemma 5b* and *Lemma 8* that in this case U_n(b) converges to a single point for any α > 0. For negative values of α we can analyze the complement sequence evaluated for α' = -α. However, **Cpl**(b) ≡ {$\underline{10}$} = {1$\underline{01}$} = {1b} so that, by *Lemma 6*, U_n(**Cpl**(b)) also converges for any positive α'. It follows that U_n(b) converges to a single point for *any* α and, in virtue of *Lemma 6*, the same applies to all sequences which contain {$\underline{01}$} as a trailer.

(c) There are many sequences which converge to an x-independent value for any α ≥ 0. For example, in virtue of *Lemma 4b, Corollary 2* and of *Lemma 6*, this category includes all sequences which contain periodic or a-periodic trailers composed of the fragments {011} and {010}. Likewise, in virtue of *Lemma 5f*, all sequences containing a trailer composed only of fragments {01^n} belong to this class.

What emerges is the following picture:

Each infinite binary sequence b defines a region C_b of α-values for which the sequence $\{\mathbf{S}_n(b)|x\}$ converges to an x-independent limit. By *Theorem I*, C_b certainly includes C_0 but, depending upon b, it may be considerably more extensive; we have seen cases in which it includes all real numbers. When C_b does not include all real numbers and $\alpha\notin C_b$, the sequence $\{\mathbf{S}_n(b)|x\}$ may still converge to a unique element of R but the limit is no longer the same for every $x\in R$ (a similar situation has been discussed also in the context of hyperpower functions [4]) Alternatively, the sequence may have a set $A_b(\alpha)$ of multiple accumulation points and oscillate between their neighborhoods. When α enters C_b, however, all the elements of $A_b(\alpha)$ coalesce into a single one and *Theorem I* assures us that

i) when $\alpha\in C_0$ then all the sets $A_b(\alpha)$ are coalesced and

ii) C_0 includes all values α for which such a simultaneous coalescence can happen.

The theorem could be therefore re-stated simply as $C_0 = \cap_b C_b$.

This discussion opens a number of interesting problems. For example, one might like to delimit the subset B_t of all *totally convergent* sequences $b\in B$ for which C_b covers all real numbers. Another open problem is the maximum cardinality of the sets $A_b(\alpha)$. Such points, however, exceed the scope of this paper and will be tackled elsewhere.

V. The BK bijection and its properties

Theorem I legitimates the following **definition** of a mapping $\mathbf{K}:B\rightarrow R$, applicable for any $\alpha\in C_0$

(V.1) $\mathbf{K}(b) = \cap_n U_{n,b} \equiv \cap_n \mathbf{M}_{n,b}([0,\infty))$.

In the rest of this paper we will assume that $\alpha\in C_0' \equiv [-e,-e^{-1})\cup(e^{-1},e]$, excluding the trivial case of $\alpha = 0$. Considering the equivalence (III.3) between inversion of the sign of α and complementing the sequences, we actually need to consider only values of $\alpha\in(e^{-1},e]$ which include bases β from $e^{1/e} = 1.444667...$ to $e^e = 15.154262....$

We now have a mapping $\mathbf{B}(r):R\rightarrow B$ which associates a binary sequence to every element r of R and a mapping $\mathbf{K}(b):B\rightarrow R$ which associates an element of R to every binary sequence b. We must still try and establish a relation between them. The task has been partially tackled in *Lemma 2g* which indicates that the two mappings might be an inverse of each other. It also points out a potential problem with some particular sequences which needs to be dealt with first.

Consider the sequence $\{0\underline{1}...\}$ which maps through $\mathbf{K}$ onto the real number 0. Since $\mathbf{E}_+(0)$ and $\mathbf{E}_-(0)$ both equal 1, the sequences $t_{00\underline{1}} \equiv \{00\underline{1}\}$ and $t_{10\underline{1}} \equiv \{10\underline{1}\}$ both map onto the real number 1 while, according to the definition (II.6), $\mathbf{B}(1)$ is unique and coincides with $t_{10\underline{1}}$. A similar situation arises with any pair of sequences containing a common starting section followed by the *trailer* $t_{00\underline{1}}$ in one case and $t_{10\underline{1}}$ in the other - they both map through $\mathbf{K}$ onto the same real number which, however, maps back through $\mathbf{B}$ uniquely onto the sequence having $t_{10\underline{1}}$ as its trailer.

In order to remove the uncertainty and pave the way for a bijection, we simply remove from B the redundant binary sequences containing the trailer $t_{00\underline{1}}$. The resulting reduced set, denoted as B^*, is the only one with which we shall henceforth concern ourselves. Notice that the mapping $\mathbf{B}:R\rightarrow B$ of (II.6) is actually of the type $\mathbf{B}:R\rightarrow B^*$ since it never maps any element of R onto one of the excluded sequences.

There is a close relation between this image-set reduction process and what one does in binary power-series representations of real numbers. There, in order to achieve a biunivocal correspondence, the trailing sequence $0\underline{1}$ of binary digits has to be excluded from the representation space in favor of the equivalent trailer $1\underline{0}$.

Lemma 9. When $\alpha \in C_0'$, the mapping $\mathbf{K}:B^* \to R$ is one-to-one, i.e., if $b,b' \in B^*$ and $b \neq b'$ then $\mathbf{K}(b) \neq \mathbf{K}(b')$.

Proof. With no loss of generality, we assume $\alpha \in (e^{-1},e]$. Suppose that the first difference between b and b' occurs at the m-*th* digit and that b_m="0" and b'_m="1". Then the nested intervals $U_{n,b} \equiv \mathbf{M}_{n,b}(U_{0\infty})$ and $U_{n,b'} \equiv \mathbf{M}_{n,b'}(U_{0\infty})$ are the same for any n<m. For n = m, however, we obtain two distinct intervals, due to the fact that $U_{m,b} \equiv \mathbf{M}_{m,b}(U_{0\infty}) = \mathbf{M}_{m-1,b}(\mathbf{E}^-(U_{0\infty})) = \mathbf{M}_{m-1,b}(U_{01})$, $U_{m,b'} \equiv \mathbf{M}_{m,b'}(U_{0\infty}) = \mathbf{M}_{m-1,b'}(\mathbf{E}^+(U_{0\infty})) = \mathbf{M}_{m-1,b}(U_{1\infty})$ and the mapping $\mathbf{M}_{m-1,b}(x)$ is continuous and monotonous (*Lemma 2a,f*). The intervals $U_{m,b}$ and $U_{m,b'}$ have one point in common, namely the image $\mathbf{M}_{m-1,b}(1)$ of 1. According to *Lemma 3*, $U_{n,b} \subset U_{m,b}$ and $U_{n,b'} \subset U_{m,b'}$ for any n > m. Consequently, two possible situations can arise:
a) either $K(b) \neq K(b')$, in which case the proof is finished, or
b) the two trailing sequences $\mathbf{T}_{m+1}(b)$ and $\mathbf{T}_{m+1}(b')$ both correspond to the real number 0. If that were not the case, in fact, there would exist an M > m such that $b_M \neq b'_M$. This would make the intervals $U_{n,b}$ and $U_{n,b'}$ disjoint for any $n \geq M$ and, consequently, we would have again $K(b) \neq K(b')$.
Case (b), however, implies that $b_{m+1} = b'_{m+1} = 0$ and $b_{m+k} = b'_{m+k} = 1$ for any k>1, which is impossible since then the sequence b would belong to the excluded ones, contradicting the assumption $b \in B^*$.

Lemma 10. When $\alpha \in C_0'$, the mapping $\mathbf{K}:B^* \to R$ is an injection, i.e., given any element $x \in R$, there exists an element $b \in B^*$ such that $\mathbf{K}(b) = x$. Moreover, $\mathbf{K}$ is an inverse of $\mathbf{B}:R \to B^*$, i.e., $\mathbf{K}(\mathbf{B}(x)) = x$.

Proof: We assume again that $\alpha \in (e^{-1},e]$. Using (II.6), construct the unique sequence $b \equiv \mathbf{B}(r) \in B^*$. Now, for any natural n, $\mathbf{M}_{n,b}$ maps $U_{0\infty}$ onto the interval $U_{n,b} = \mathbf{M}_{n,b}(U_{0\infty})$. The n-*th* descendant of x, defined by (II.5) as $\mathbf{L}^n(x)$, certainly belongs to $U_{0\infty}$ so that its image under $\mathbf{M}_{n,b}(\mathbf{L}^n(x))$ belongs to $U_{n,b}$. Since the mapping $\mathbf{M}_{n,b}:U_{0\infty} \to U_{n,b}$ is a bijection with the inverse $\mathbf{L}^n$ (*Lemma 2c*), we have $\mathbf{M}_{n,b}(\mathbf{L}^n(x)) = x$ and therefore $x \in U_{n,b}$ for any n. Consequently $x \in \cap_n U_{n,b}$. But, according to *Theorem I*, $\cap_n U_{n,b}$ contains exactly one element of *R*, which, by the definition (V.1), is $\mathbf{K}(b)$. Hence $\mathbf{K}(b) = \cap_n U_{n,b} = x$.

Put together, the last two lemmas amount to

Theorem II.
When $\alpha \in [-e,-e^{-1}) \cup (e^{-1},e]$, the mappings $\mathbf{K}:B^* \to R$ and $\mathbf{B}:R \to B^*$ are bijections and inverses of each other.

We shall henceforth refer to the pair of mappings **B** and **K** as *the BK bijection*. In the remainder of this Section we shall try and establish some of its basic properties and discuss a potential application.

The next lemma shows that any real number r can be approximated by means of finite starting sections $\mathbf{S}_n(\mathbf{B}(r))$ of its corresponding binary sequence $\mathbf{B}(r)$. It also shows that the metric "weight" of an element b_k of a binary sequence decreases with increasing index k.

Lemma 11. Given a binary sequence $b \in B^*$ corresponding to $r = \mathbf{K}(b)$ and a positive number ε, there exists an integer n such that any sequence $b' \in B^*$ such that $\mathbf{S}_n(b') = \mathbf{S}_n(b)$ corresponds to a real number $r' = \mathbf{K}(b')$ which differs from r by less than ε.

Proof: By Theorem I, $\lim_{n\to\infty}\rho(U_{n,b}) \equiv \lim_{n\to\infty} \rho(\mathbf{M}_{n,b}(U_{0\infty})) = 0$. Consequently, there exists an n_ε such that $\rho(U_{n,b}) < \varepsilon$ for any $n > n_\varepsilon$. Since any sequence B^* with the first n_ε elements identical to those of b maps $U_{0\infty}$ into $U_{n,b}$, it follows that $\mathbf{K}(b)$ and $\mathbf{K}(b')$ both belong to $U_{n,b}$ and therefore $|r-r'| < \varepsilon$.

Considering the way the BK bijection was constructed, the following Lemma is nearly trivial. However, it is conceptually important since it establishes the possibility of carrying out a number of mathematical operations on *R* by means of isomorphic mappings in *B**.

Lemma 12. Let b∈B^* and x = **K**(b). Then
(a) $\mathbf{L}^n(x) \equiv \mathbf{L}^n(\mathbf{K}(b)) = \mathbf{K}(\mathbf{T}_n(b))$.
Special case for n=1: $\mathrm{abs}(\ln_\beta(x)) = \mathbf{K}(\mathbf{T}_1(b))$.
(b) {c|x} ≡ {c|**K**(b)} = **K**(cb) for any *finite* binary sequence c.
Special cases: exp(αx) = **K**(1b) and exp(-αx) = **K**(0b).
(c) $x^{-1} = \mathbf{K}(\mathbf{Inv}(b))$.

Proofs.

Statement (a): Let b' = $\mathbf{T}_n$(b). By Theorem I, $U_{b'}$ contains a single element of *R* which is **K**(b') = **K**($\mathbf{T}_n$(b)) and, by *Lemma 6*, x = **K**(b) = $\mathbf{M}_{n,b}$(**K**(b')) = $\mathbf{M}_{n,b}$(**K**($\mathbf{T}_n$(b))). By *Lemma 2c*, $\mathbf{L}^n$ is the inverse mapping of $\mathbf{M}_{n,b}$ on its image set (which contains **K**(b)). Hence, applying $\mathbf{L}^n$ to both sides of the equation, we obtain the result.

Statement (b): Let n = len(c). Then $\mathbf{T}_n$(cb) = b and $\mathbf{M}_{n,cb}$(x) = {c|x}. Applying again *Lemma 6*, one obtains **K**(cb) = $\mathbf{M}_{n,cb}$(**K**($\mathbf{T}_n$(cb))) = $\mathbf{M}_{n,cb}$(**K**(b)) = {c|x}.

Statement (c): Let b = {0$\mathbf{T}_1$(b)}. Then **Inv**(b) = {1$\mathbf{T}_1$(b)} and, applying *Statement (b)*, **K**(**Inv**(b)) = **K**(1$\mathbf{T}_1$(b)) = {1|**K**($\mathbf{T}_1$(b))} = 1/{0|**K**($\mathbf{T}_1$(b))} = 1/**K**(b) = 1/x. Similarly, in the case of b = {1$\mathbf{T}_1$(b)} we have **K**(**Inv**(b)) = **K**(0$\mathbf{T}_1$(b)) = {0|**K**($\mathbf{T}_1$(b))} = 1/{1|**K**($\mathbf{T}_1$(b))} = 1/**K**(b) = 1/x.

The ease with which it is possible to determine the binary sequence corresponding to x^{-1} from the one for x, one naturally wonders whether there might also exist simple algorithms for the binary operations in B^* isomorphic with the four basic binary arithmetic operations in *R*. Unfortunately, what is presently lacking are efficient algorithms for the sum and difference of two non-negative reals represented by their images in B^*. In other words, given two sequences a,b∈B^*, we don't know how to efficiently compute the sequences s,d∈B^* such that **K**(s) = **K**(a)+**K**(b) and, assuming **K**(a) ≥ **K**(b), **K**(d) = **K**(a)-**K**(b). Were this possible, multiplication and division of **K**(a) and **K**(b) would become trivial because, according to (III.4) and *Lemma 6*, they reduce to summing/subtracting the first descendants of the two numbers, i.e., of **K**($\mathbf{T}_1$(a)) and **K**($\mathbf{T}_1$(b)).

The following lemma introduces a strict, total ordering of the set B^* which is isomorphic with the natural ordering of real numbers. Remembering that zer($\mathbf{S}_n$(b)) is the number of "0" elements among the first n ones of b, the ordering "recipe" turns out to be surprisingly simple.

Lemma 13. Assume α∈(e^{-1},e] and let b,b'∈B^* be two binary sequences such that their first n elements are identical, while b_n= "0" and b'_n= "1". Then **K**(b) < **K**(b') when zer($\mathbf{S}_n$(b)) is even and **K**(b) > **K**(b') when it is odd.

Proof. **K**(b) ∈ $\mathbf{M}_{n+1,b}(U_{0\infty}) = \mathbf{M}_{n,b}(U_{01})$ and **K**(b') ∈ $\mathbf{M}_{n+1,b'}(U_{0\infty}) = \mathbf{M}_{n,b'}(U_{1\infty}) \equiv \mathbf{M}_{n,b}(U_{1\infty})$. Since all elements of U_{01} are smaller than any element of $U_{1\infty}$, the statement follows from *Lemma 2f*.

Definition. For α∈(e^{-1},e], the ordering of B^* isomorphic with that of *R* can be therefore defined as follows:
Let b,b'∈B^* be two *distinct* binary sequences such that their first n elements are identical and b_n= "0", b'_n= "1". Then we set b < b' when zer($\mathbf{S}_n$(b)) is even and b > b' when it is odd. In the case of α∈[-e,-e^{-1}), the definition needs to be modified by considering the complement sequences **Cpl**(b) and **Cpl**(b') instead of b and b' themselves (alternatively, one can replace the zero count zer($\mathbf{S}_n$(b)) by the count of "1"s, i.e., len($\mathbf{S}_n$(b)) - zer($\mathbf{S}_n$(b))).

Extensions of K(b) and related functions.

For any fixed sequence b∈*B*, the mapping $\Theta_b(\alpha) \equiv \mathbf{K}_\alpha(b)$ can be intended as a function of real argument α. *Theorem I* proves that, within the domain α∈C_0, the function $\Theta_b(\alpha)$ is always well-defined, regardless of the choice of b. In general, however, the definition of $\Theta_b(\alpha)$ can be extended to a broader domain C_b within which (V.1) still defines a unique element of *R*. For example, we have seen in the preceding Section that any sequence of the type {b$\underline{01}$} with finite {b} admits an extension of $\Theta_b(\alpha)$ to any real value of α.

Other types of extensions become possible by relaxing the definition (V.1). For example, one can choose a particular value $x_0 \in R$ and define $\Theta'_b(\alpha) \equiv \mathbf{K'}(b) = \lim_{n\to\infty} \mathbf{M}_{n,b}(x_0)$, provided that the limit exists. That this is indeed an extension is evident from the fact that whenever $\Theta_b(\alpha)$ exists, $\Theta'_b(\alpha)$exists as well and $\Theta_b(\alpha) = \Theta'_b(\alpha)$. The functions $\Theta'_b(\alpha)$ include as a special case the classical hyperpower function [1] which corresponds to $x_0 = 1$, $b = \{\underline{1}\}$ and converges for any $\alpha \geq -e$, though the limit is finite only when $-e \leq \alpha < e^{-1}$.

VI. Representation of real numbers by means of bips

One of the potential applications of the BK bijection is the representation of real numbers by means of the binary sequences **B**(x). The fact that such sequences are composed contain just two elements "0" and "1" makes them particularly suitable for machine implementation where each element corresponds to a memory bit.

The most important feature of such a representation - just after the essential requirement that **B** and **K** be bijections and inverses of each other (*Theorem II*) - is the fact that finite sections of **B**(x) of increasing length provide a progressively improving approximation to the represented real number x (*Lemma 11*) and that the convergence turns out to be quite good as illustrated by the example in Table I.

The resulting *binary iterated-powers representation* (BIPR) is a conceptually new approach to the problem of representing real numbers, not just an implementation variation of an existing one. For example, all the conventional representations such as IEEE 754 [5,6] with fixed or variable-width exponent, logarithmic number systems [7], level-index [8] and symmetric level-index [8,9] fall into a qualitatively different group based on variations of what we shall call the *power-series representation* (PSR or, in its binary version, BPSR).

BIPR is also qualitatively different from representations based on sequences of functions [10], continued fractions [11] and infinite compositions of linear fractional transformations [12,13]. Compared with such approaches, BIPR appears to be considerably more intuitive and straightforward (though this does not automatically imply a superior computational efficiency).

Any real numbers representation should possess a number of properties against which different representations can be checked and compared. Deferring a detailed discussion of such properties to a forthcoming paper, let us list here just a few advantages and disadvantages of BIPR with respect to BPSR.

For numbers of reasonable size, say between 10^{-100} to 10^{+100}, BIPR is very *efficient* and requires a number of bits comparable to the BPSR in its IEEE-754 version. Compared to BPSR, however, BIPR has a staggeringly wider range. In the neperian base (α=1), a starting segment of the type {1111} brings us close to 10^{+148} and adding just one more "1" extends that to a value whose decimal exponent has 148 digits. The magnitude of any number corresponding to a bip with, say, 32 consecutive "1"s is beyond imagination [14,15]. The same applies to the smallest number representable by a bip of a certain length. According to *Lemma 12c*, in fact, the inverse of any bip can be obtained simply by inverting the first element of the binary sequence. Thus, for example, "01111" encodes a real of the order of 10^{-148}.

A BIPR code for real numbers would not need any separate exponent section and, in addition, it would spell a definitive demise of the concepts of overflow and underflow. In any practical representation the preferred base would actually be smaller than e since there is hardly any need for such a steep growth of range with the number of employed bits. The two most attractive choices seem to be $\beta = 2$ with a growth rate comparable to that of Fermat numbers $F_n = 1+\{1^n|1\}_{\beta=2}$ and $\alpha = 4/e^2$ which, according to the hints in *Lemma 5*, might entail a few theoretical advantages.

Among other advantages of BIPR is the simplicity of comparing the size of two numbers (*Lemma 13*). The recipe for doing so in BIPR is only marginally less efficient than in pure BPSR and, due to the absence of an exponent, better than in any practical BPSR implementation, including IEEE-754. Even more striking is the extreme simplicity of computing the inverse of a number (*Lemma 12c*), or its logarithm and exponential (*Lemma 12a,b*) - tasks which in BPSR are anything but trivial.

For what regards basic arithmetic, multiplication and division would become trivial if only we had an efficient algorithm for BIPR sum and difference. Despite recent attempts [16] to tackle a closely related task, this problem is at present largely unsolved. The situation contrasts sharply with that of BPSR where addition and subtraction are very simple, multiplication is moderately difficult and division is rather demanding.

Comparing the advantages and disadvantages of BIPR and BPSR, it is likely that, should someone discover an efficient algorithm for addition and subtraction in BIPR, the latter might supersede all the presently ubiquitous PBSR implementations. Regardless of the final outcome, however, the very existence of two distinct real number representations with dramatically different characteristics of the associated algorithmic machinery is quite stimulating.

VII. Conclusions

We have proved that, within a range C_0 of values of the base β, the *bips* $\{\mathbf{S}_n(b|x\}$ converge for every infinite binary sequence b to a limit which depends only on the sequence b and not on the value of its starting argument x. Moreover, for a subset B^* of infinite binary sequences, the resulting mapping $\mathbf{K}:B^*\to R$ turns out to be a bijection whose inverse $\mathbf{B}:R\to B^*$ is explicitly defined.

While establishing these results, we have derived a number of explicit identities and inequalities for finite bips, exploiting an ad-hoc notation which simplifies their mathematical treatment. Incidentally, we have also seen that the bips and their derivatives are computationally easily manageable.

The *BK-bijection* has a number of interesting properties, some of which have been analyzed. As usual, the topic opens many more problems than those which it settles. This regards, in particular, the behavior of the **K** mapping for various binary sequences b and for base values which lie outside the universal-convergence domain C_0 . There are also categories of related functions which appear to merit further inquiry.

Some of the considerations on the use of the BK-bijection to represent real numbers might eventually lead to establishing a general *real-numbers representation theory*. This topic shall be tackled in more detail in a forthcoming paper.

This study actually started from the observation that it is impossible to reliably compute the result of iterated applications of the function abs(ln(x)) when the number of required iterations exceeds a certain limit. Computing the iterated values amounts to estimating the **L**-progeny sequence (II.5) of x. The **L**-progeny sequences which one obtains in practice when using various floating point (FP) implementations start grossly mismatching each other after a relatively modest number of steps. Thus, for example, comparing standard single-precision and double-precision values of $\mathbf{L}^n(2)$, the mismatch is total after about 30 steps. Even using the same double-precision IEEE-754 format but slightly different algorithms (such as the hard-wired Intel FP processor versus an FP emulator), the mismatch becomes total after about 62 steps.

The reason for this behavior is now quite clear. Since the **L**-progeny sequence *defines* the binary sequence **B**(x), pretending correct results for an ever increasing number of steps amounts to determining the value of x with a precision which would eventually exceed the precision we have started with. This being impossible, the **L**-progeny sequence members x_n and the elements of **B**(x) *must* at some point become meaningless. Due to the fact that the convergence rate of BIPR is only slightly inferior to that of BPSR, this happens after a number of steps comparable to the number of BPSR bits used to represent the number.

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